PROBLEM SET 2: SOME RANDOM PROBLEMS

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Choose 3 problems and hand them in next week, Wednesday January 22.

**Problem 1. *** There are 25 matches on the table. Two players take turns. Each turn they have to take away 1, 2 or 3 matches. The person taking the last match loses. Show that the second player always can win this game.


**Problem 2. ***(Hungarian Eötvös competition) Find the sum of all four digit numbers which contain that contain only the digits 1, 2, 3, 4, 5, each at most once.

**Problem 3. ** A man has 10 pockets and 44 silver dollars. He wants to put his dollars into his pocket so distributed that each pocket contains a different number of dollars.

(a). Can he do so?
(b). Generalize the problem considering $p$ pockets and $n$ dollars. The problem is most interesting when

$$n = \frac{(p + 1)(p - 2)}{2}.$$  

Why?

**Problem 4. **** A bacterium splits into two identical ones with probability $p$, otherwise it dies. What is the probability that, beginning with this one bacterium, the colony lasts forever?

**Problem 5. ** What is the maximum and the minimum value of

$$\frac{ab + bc + ca}{a^2 + b^2 + c^2}$$

where $a, b, c$ are real numbers, not all equal to 1. For which $a, b, c$ do we get this maximum or minimum value?
Problem 6. ** Suppose we have a scale (one with two arms which can only decide if two weights are equal or which one is heavier). There are 9 coins, of which one is counterfeit. All coins weight the same, except the counterfeit one, which is heavier. Show that weighing only twice, one can determine which coin is counterfeit.

Problem 7. *** Prove that every polygon of perimeter $2a$ can be completely covered by a circular disc of diameter $a$.

Problem 8. *** Let $P(x)$ the the unique polynomial of degree $n$ such that
\[ P(x) = \frac{x}{x + 1}, \quad \text{for } x = 0, 1, 2, \ldots, n \]
Determine $P(n + 1)$.

Problem 9. *** Five points lie in the unit square. Show that the distance between two of them is at most $\frac{1}{2}\sqrt{2}$.

Problem 10. **** If $a_0, a_1, \ldots, a_n$ are real numbers satisfying
\[ \frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n + 1} = 0 \]
Show that the polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ has at least one real root.

Problem 11. ***** Evaluate
\[ \sum_{n=2}^{\infty} \frac{\varphi(n)}{2^n - 1} \]
where $\varphi(n)$ is Euler’s function, so $\varphi(n)$ is the number of integers $m$ with $1 \leq m \leq n$ and $m$ and $n$ are relatively prime.