PROBLEM SET 5: MORE RANDOM PROBLEMS

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Choose 3 problems and hand them in next week, Wednesday February 12. January 22.

Problem 1. *** The number 3 can be expressed as a sum of one or more positive integers, taking order into account, in four ways, namely, 3, 1+2, 2+1 and 1+1+1. Give and prove a formula for the number of ways that n can be expressed as a sum of one or more positive integers.

Problem 2. *** Find an n and positive integers $a_1, a_2, \ldots, a_n$ such that $a_1 + a_2 + \cdots + a_n = 2003$ and $a_1 a_2 \cdots a_n$ is maximal.

Problem 3. ** Among n people at a party, show that there are two who know exactly the same number of people (present at the party).

Problem 4. **** Let $f : [0, 1] \to \mathbb{R}$ be continuous and suppose that $f(0) = f(1)$. Prove that for each positive integer $n$ there is an $x$ in $[0, 1 - \frac{1}{n}]$ such that $f(x) = f(x + 1/n)$.

***Prove that the following sequence converges and find the limit:

$$\sqrt{1}, \sqrt{1 + \sqrt{1}}, \sqrt{1 + \sqrt{1 + \sqrt{1}}}, \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}, \ldots$$

Problem 5. *** Prove that there are infinitely many prime numbers of the form $4n - 1$ where $n$ is a positive integer.

Problem 6. ** Show that every positive integer has a multiple whose decimal representation involves all ten digits.

Problem 7. ** Suppose $X$ is a finite nonempty set. Let $f$ be a function from $X$ to itself. Show that there exists a finite nonempty subset $S \subseteq X$ such that $f(S) = S$.

Problem 8. ****(Putnam) Given $n$ points on the sphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, demonstrate that the sum of the squares of the distances between them does not exceed $n^2$.

Problem 9. *** (Putnam) Prove

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} \, dx.$$
Problem 10. ******* Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be any map with the following property: Whenever \( P \) and \( Q \) are two points in the plane with distance \( d(P, Q) = 1 \) then their images also have distance 1 (so \( d(f(P), f(Q)) = 1 \)). Prove that \( f \) must be an isometry (which means that \( d(f(P), f(Q)) = d(P, Q) \) for all \( P, Q \in \mathbb{R}^2 \)).