Exam 1

Math 425, Section 3

Instructor: Harm Derksen

(50 Minutes)

Monday, February 7, 2005

Name:

Work on the space provided (you may use the back). No books allowed. Calculators may be used. Show your work!

Circle your final answer!

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(1) We roll two fair dice. $E$ is the event that the sum of the dice is 6. $F$ is the event that the sum of the dice is 7. $G$ is the event that the second die is a 1. (Motivate your answer using the definition of independence in the questions below.)

(a) (5 pts) Are $E$ and $F$ independent events?

No. $P(E) = \frac{5}{36}$, $P(F) = \frac{6}{36} = \frac{1}{6}$ and $P(EF) = 0 \neq P(E)P(F)$.

(b) (5 pts) Are $E$ and $G$ independent events?

No. $P(E) = \frac{5}{36}$, $P(G) = \frac{1}{6}$ and $P(EG) = \frac{1}{36}$ (because $EG = \{(5, 1)\}$). So $P(EG) \neq P(E)P(G)$.

(c) (5 pts) Are $F$ and $G$ independent events?

Yes. $P(F) = \frac{1}{6}$, $P(G) = \frac{1}{6}$ and $P(FG) = \frac{1}{36}$. So $P(FG) = P(F)P(G)$. 
(2) If a student is sick (s)he will go to school with probability 0.1. If a student is not sick (s)he will go to school with probability 0.9. A randomly chosen student is sick with probability 0.2.

(a) (10 pts) What is the probability that a randomly chosen student does not to school?

\[ F = \{\text{student is sick}\}, \ E = \{\text{student goes to school}\}. \ P(F) = 0.2, \ P(E|F) = 0.1, \ P(E|F^c) = 0.9. \text{ Use Bayes formula:} \]

\[ P(E) = P(E : F)P(F) + P(E|F^c)P(F^c) = \]

\[ = (0.1)(0.2) + (0.9)(0.8) = 0.02 + 0.72 = 0.74. \]

So

\[ P(E^c) = 1 - P(E) = 1 - 0.74 = 0.26. \]

(b) (10 pts) What is the probability that a randomly chosen absent student is sick?

\[ P(E^c F) = P(E^c|F)P(F) = (1 - P(E|F))P(F) = (1 - 0.1)(0.2) = 0.18 \]

and \[ P(E^c) = 1 - P(E) = 1 - 0.74 = 0.26. \text{ So} \]

\[ P(F|E^c) = \frac{P(E^c F)}{P(E^c)} = \frac{0.18}{0.26} \approx 0.692. \]
(3) (15 pts) How many ways are there to form 4 soccer teams out of 44 soccer players, where each team must have 11 players?

\[
\binom{44}{11,11,11,11} \frac{4!}{4!} = 43627992869961630486720
\]

(4) (15 pts) A ballot box contains 10 ballots for a republican candidate, and 9 ballots for a democrat (and no other ballots). We take 7 ballots out of the box at random. What is the probability that exactly 3 of them were cast for the republican candidate?

\[
\frac{\binom{10}{3} \binom{9}{4}}{\binom{19}{7}} = \frac{1260}{4199} \approx 0.300
\]
(5) (15 pts) How many ways are there to arrange \(A, B, C, D, E, F, G, H\) such that \(A\) and \(B\) are next to each other and \(C\) appears before \(D\) and \(E\)?

There are 7! ways to arrange, \(X, C, D, E, F, G, H\), where we replace \(X\) by \(AB\) or \(BA\) (2 possibilities). One third of these arrangements will have \(C\) before \(D\) and \(E\). So the answer is

\[
\frac{7! \cdot 2}{3} = 3360.
\]

(6) (20 pts) Alice, Bob and Carl each have 3 keys to their own apartment. They mix the nine keys and they cannot tell which key is which. Then each person takes one key and goes home. What is the probability that at least 1 person can unlock their apartment door?

Let \(A, B, C\) be the events that Alice, Bob, Carl (respectively) can unlock their apartment door. We have

\[
P(A) = P(B) = P(C) = \frac{1}{3},
\]

\[
P(AC) = P(BC) = P(AB) = (B|A)P(A) = \frac{3}{8} \cdot \frac{1}{3} = \frac{1}{8},
\]

and

\[
P(ABC) = P(C|AB)P(AB) = \frac{3}{7} \cdot \frac{1}{8} = \frac{3}{56}.
\]

Therefore

\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC) = 3 \cdot \frac{1}{3} - 3 \cdot \frac{1}{8} + \frac{3}{56} - \frac{56}{56} + \frac{56}{56} + \frac{56}{56} = \frac{38}{56} = \frac{19}{28} \approx 0.679.
\]