Exam 2
Math 425, Section 3

Instructor: Harm Derksen

(50 Minutes)

Friday, March 18, 2005

Name:

Work on the space provided. No books or notes allowed. Calculators may be used. Show your work!

Circle your final answer!
(1) A computer help-line get on average 2 calls between noon and 1pm. Let $X$ be the number of calls between noon and 1pm on the help-line on a given day.

(a) (5 pts) What type of distribution (Bernoulli/binomial/Poisson/geometric/negative binomial/hypergeometric/uniform) gives the best approximation for $X$?

Poisson, $\lambda = 2$.

(b) (10 pts) What is the probability that $X \geq 4$?

$$1 - P(0) - P(1) - P(2) - P(3) = 1 - e^{-2} - 2e^{-2} - \frac{2^2}{2}e^{-2} - \frac{2^3}{6}e^{-2} =$$

$$= 1 - \frac{19}{3}e^{-2} \approx 0.1429.$$
(2) Anna and Bert play a game. In each game, Anna pays Bert $2.50 and she rolls two dice. If the smallest of the dice is \( i \), then Bert pays \( i \) dollars to Anna.

(a) (15 pts) What is the expected gain of Anna?

\[
(-1.5)p(-1.5) + (-0.5)p(-0.5) + (0.5)p(0.5) + (1.5)p(1.5) + (2.5)p(2.5) + (3.5)p(3.5) = \\
(-1.5)\frac{11}{36} + (-0.5)\frac{9}{36} + (0.5)\frac{7}{36} + (1.5)\frac{5}{36} + (2.5)\frac{3}{36} + (3.5)\frac{1}{36} = \frac{1}{36} \approx 0.0278
\]

(b) (10 pts) What is the expected number of games that Anna and Bert have to play until Anna rolls at least one 1 (in which case she loses $1.50)?

Let \( Y \) be the number of games needed. Then \( Y \) has a geometric distribution with \( p = P(\text{Anna rolls at least one 1}) = \frac{11}{36} \). Therefore

\[
EY = \frac{1}{p} = \frac{36}{11} \approx 3.273.
\]
(3) \( X \) is a continuous random variable whose density function is given by
\[
f(x) = \begin{cases} 
0 & \text{if } x < 0 \text{ or } x \geq 3; \\
1/5 & \text{if } 0 \leq x < 1; \\
2/5 & \text{if } 1 \leq x < 3.
\end{cases}
\]

(a) (10 pts) Determine \( P(X < 2) \).
\[
P(X < 2) = \int_{-\infty}^{2} f(x) \, dx = \int_{0}^{1} \frac{1}{5} \, dx + \int_{1}^{2} \frac{2}{5} \, dx = \frac{1}{5} + \frac{2}{5} = \frac{3}{5} = 0.6
\]

(b) (10 pts) Compute \( E(X) \).
\[
E(X) = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{0}^{1} \frac{1}{5} x \, dx + \int_{1}^{3} \frac{2}{5} x \, dx = \\
\left[ \frac{1}{10} x^2 \right]_{0}^{1} + \left[ \frac{1}{5} x^2 \right]_{1}^{3} = \frac{1}{10} + \frac{8}{5} = 1.7
\]

(c) (10 pts) Compute \( \text{Var}(X) \).
\[
E(X^2) = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{0}^{1} \frac{1}{5} x^2 \, dx + \int_{1}^{3} \frac{2}{5} x^2 \, dx = \\
\left[ \frac{1}{15} x^3 \right]_{0}^{1} + \left[ \frac{2}{15} x^3 \right]_{1}^{3} = \frac{1}{15} + \frac{52}{15} = \frac{53}{15}
\]
\[
\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{53}{15} - \left( \frac{17}{10} \right)^2 = \frac{193}{300} \approx 0.6433.
\]
(4) (10 pts) The probability of having the rare disease *mathematosis* is 1 in 100,000. What is the probability that a city with 50,000 inhabitants has at least 2 people with this severe disease? For this question you **must** use an approximation with a Poisson random variable.

We approximate with a Poisson RV with $\lambda = np = 50,000 \cdot (1/100,000) = \frac{1}{2}$.

$$P(X \geq 2) = 1 - P(0) - P(1) = 1 - e^{-1/2} - \frac{1}{2} e^{-1/2} = 1 - \frac{3}{2} e^{-1/2} \approx 0.0902.$$ 

(5) (20 pts) A teacher has 10 exams in his drawer. 7 of the exams have 10 a/b multiple choice questions. The passing score for these exams is 8. The other 3 exams have 8 a/b/c multiple choice questions and have a passing score of 5. A student comes into the office. The teacher draws a random exam from his drawer. What is the probability that the student passes the exam if (s)he guesses each question?

*a/b exams: binomial with p = \frac{1}{2}, n = 10:*

$$P(X \geq 7) = P(7) + P(8) + P(9) + P(10) =$$

$$= \binom{10}{7} \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + \binom{10}{8} \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + \binom{10}{9} \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 + \binom{10}{10} \left(\frac{1}{2}\right)^{10} = \frac{11}{64} \approx 0.1719$$

*a/b/c exams: binomial with p = \frac{1}{3}, n = 8:*

$$P(X \geq 5) = P(5) + P(6) + P(7) + P(8) = \binom{8}{5} \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^3 + \binom{8}{6} \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^2 +$$

$$+ \binom{8}{7} \left(\frac{2}{3}\right)^7 \left(\frac{1}{3}\right)^1 + \binom{8}{8} \left(\frac{2}{3}\right)^8 \approx \frac{577}{6561} \approx 0.088$$

**Bayes formula:**

$$P(\text{pass}) = (0.7)(0.1719) + (0.3)(0.088) = 0.147.$$
### Discrete Random Variables

<table>
<thead>
<tr>
<th>distribution</th>
<th>parameters</th>
<th>( P(X = i) )</th>
<th>( \text{E}(X) )</th>
<th>( \text{Var}(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernoulli</td>
<td>( p )</td>
<td>( p(0) = 1 - p, p(1) = p )</td>
<td>( p )</td>
<td>( p(1 - p) )</td>
</tr>
<tr>
<td>Binomial</td>
<td>( n, p )</td>
<td>( \binom{n}{i} p^i (1 - p)^{n-i} )</td>
<td>( np )</td>
<td>( np(1 - p) )</td>
</tr>
<tr>
<td>Poisson</td>
<td>( \lambda )</td>
<td>( \lambda^i \frac{e^{-\lambda}}{i!} )</td>
<td>( \lambda )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>Geometric</td>
<td>( p )</td>
<td>( (1 - p)^{i-1}p )</td>
<td>( \frac{1}{p} )</td>
<td>( \frac{1 - p}{p^2} )</td>
</tr>
<tr>
<td>Negative Bin.</td>
<td>( p, r )</td>
<td>( \frac{(i - 1)}{(r - 1)} p^r (1 - p)^{i-r} )</td>
<td>( \frac{r}{p} )</td>
<td>( \frac{r(1 - p)}{p^2} )</td>
</tr>
<tr>
<td>Hypergeometric</td>
<td>( N, m, n )</td>
<td>( \frac{\binom{m}{i} \binom{N-m}{n-i}}{\binom{N}{n}} )</td>
<td>( \frac{nm}{N} )</td>
<td>( \frac{(N-n)nm(N-m)}{(N-1)N^2} )</td>
</tr>
</tbody>
</table>

### Continuous Random Variables

<table>
<thead>
<tr>
<th>distribution</th>
<th>parameters</th>
<th>density function</th>
<th>( \text{E}(X) )</th>
<th>( \text{Var}(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>( \alpha, \beta )</td>
<td>( f(x) = \frac{1}{\beta - \alpha} ) if ( \alpha &lt; x &lt; \beta ), 0 otherwise</td>
<td>( \frac{1}{2}(\alpha + \beta) )</td>
<td>( \frac{1}{12}(\beta - \alpha)^2 )</td>
</tr>
<tr>
<td>Normal</td>
<td>( \mu, \sigma )</td>
<td>( \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma}} )</td>
<td>( \mu )</td>
<td>( \sigma^2 )</td>
</tr>
</tbody>
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