HOMEWORK 2, MATH 425, SECTION 3

HARM DERKSEN

Exercise 1 (Ch 2, problems, 12)**.** The sample space S has 100 elements. Let E, F, G be the events that a random student is Spanish, French, German respectively. We have $P(E) = 28/100$, $P(F) = 26/100$, $P(G) = 16/100$ $P(EF) = 12/100$, $P(EG) = 4/100$, $P(FG) = 6/100$ and $P(EFG) = 2/100$.

(a)
$$P(E^cF^cG^c) = P((E \cup F \cup G)^c) = 1 - P(E \cup F \cup G)$$
and
$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) +$$
$$+P(EFG) = \frac{28 + 26 + 16 - 12 - 4 - 6 + 2}{100} = \frac{50}{100} = \frac{1}{2} = 0.5$$
Therefore
$$P(E^cF^cG^c) = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

(b) The probability that a student takes at least 2 classes is
$$P(EF \cup FG \cup EG) = P(EF) + P(FG) + P(EG) - P((EF)(FG)) -$$
$$P((EF)(EG)) - P((FG)(EG)) + P((EF)(FG)(EG)) = P(EF) +$$
$$+P(FG) + P(EG) - 2P(EFG) = \frac{12 + 4 + 6 - 2 \cdot 2}{100} = \frac{18}{100}.$$
So the probability that a student takes exactly one class is
$$P(\geq 1) - P(\geq 2) = \frac{50 - 18}{100} = \frac{32}{100} = \frac{8}{25} = 0.32$$

(c) From $P(E \cup F \cup G) = \frac{50}{100}$ follows that $|E \cup F \cup G| = 50$. There are $\binom{100}{2}$ ways to choose 2 students and $\binom{50}{2}$ ways to choose two students who do not take any class. So
$$P(\text{both students take no class}) = \frac{\binom{50}{2}}{\binom{100}{2}} = \frac{50 \cdot 49}{100 \cdot 99} = \frac{49}{198}.$$
and
$$P(\text{at least 1 student takes a class}) = 1 - P(\text{both students take no class}) =$$
$$= 1 - \frac{49}{198} = \frac{149}{198} \approx 0.753.$$

Exercise 2 (Chapter 2, problems, #16).

(a) Sample space has size 6^5. For "no two alike" there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$ possibilities.
$$P(\text{no two alike}) = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6^5} = \frac{5}{54} \approx 0.0926.$$

(b) There are $\binom{5}{2}$ ways of choosing the 2 positions which will form the pair. Then there 6 ways of choosing the number of eyes for the pair and $5 \cdot 4 \cdot 3$ to choose the number of eyes for the other 3 dice. So
$$P(\text{one pair}) = \frac{\binom{5}{2} 6 \cdot 5 \cdot 4 \cdot 3}{6^5} = \frac{25}{54} \approx 0.4630$$

(c) There are $\binom{5}{2}$ ways to choose the first pair and $\binom{3}{2}$ ways the second pair. But the order of the pairs does not matter so we divide by 2. This means that there are
$$\frac{\binom{5}{2}\binom{3}{2}}{2}$$
ways of grouping the five positions into a two groups of 2 and one group of 1. Then there are $6 \cdot 5 \cdot 4$ ways of assigning the number of eyes to the groups. We get
$$P(\text{two pairs}) = \frac{\binom{5}{2}\binom{3}{2} 6 \cdot 5 \cdot 4}{2 \cdot 6^5} = \frac{25}{108} \approx 0.2315.$$

(d)
$$P(\text{three alike}) = \frac{\binom{5}{3} 6 \cdot 5 \cdot 4}{6^5} = \frac{25}{162} \approx 0.1543.$$

(e)
$$P(\text{full house}) = \frac{\binom{5}{3} 6 \cdot 5}{6^5} = \frac{25}{648} \approx 0.386.$$

(f)
$$P(\text{four alike}) = \frac{\binom{5}{4} 6 \cdot 5}{6^5} = \frac{25}{1296} \approx 0.0193.$$

(g)
$$P(\text{five alike}) = \frac{6}{6^5} = \frac{1}{6^4} \approx 0.0008.$$

Exercise 3 (Chapter 2, problems, #17). There are 8 rooks to be placed on a 8×8 chessboard. There are $\binom{64}{8}$ ways to do this. Two rooks can hit each other if they are in the same row or in the same column. Let's label the columns $1, 2, \ldots, 8$ and the columns $1, 2, \ldots, 8$ (usually they are labelled by $A, B, \ldots, H$). Each colunn must contain exactly 1 rook. Suppose that the rook in column i is in row k_i. Then $k_1, k_2, \ldots, k_8$ must be distinct (otherwise two rooks are in the same row), hence they form a permutation of

$1, 2, \ldots, 8$. There are $8!$ different arrangements of $1, 2, \ldots, 8$, so there are $8!$ ways to place the rooks such that no two can hit each other. The answer is

$$\frac{8!}{\binom{64}{8}} = \frac{560}{61474519} \approx 0.911 \times 10^{-5}.$$

Exercise 4 (Chapter 2, problems, #52)**.** There are $\binom{20}{8}$ ways of choosing 2 shoes.

(a) To choose 8 distinct shoes, we can first choose 8 out of the 10 pairs and then take from each of those pairs either the left shoe or the right shoe. There are $\binom{10}{8}$ ways of choosing the 8 pairs. For each of those 8 pairs we have two choices. So there are $\binom{10}{8}2^8$ ways of choosing 8 shoes without forming a pair. The probability is

$$\frac{\binom{10}{8}2^8}{\binom{20}{8}} = \frac{384}{4199} \approx 0.0915.$$

(b) We first choose 1 of the 10 pairs. After that we still have to choose 6 shoes from the remaining 9 pairs such that no new pair is formed. Similar reasoning as in (a) shows that there are $\binom{9}{6}2^6$ ways of choosing the remaining 6 shoes. So the probability is

$$\frac{10\binom{9}{6}2^6}{\binom{20}{8}} = \frac{1792}{4199} \approx 0.427.$$

Exercise 5. Enumerate the suits by $1, 2, 3, 4$ (instead of *hearts, spades, clubs, diamonds*). Let E_i the event that suit i is void. We have

$$P(E_i) = \frac{\binom{39}{13}}{\binom{52}{13}}.$$

We are asked to compute $P(E_1 \cup E_2 \cup E_3 \cup E_4)$. **If** the events E_1, E_2, E_3, E_4 were mutually exclusive then we would have

$$P(E_1 \cup E_2 \cup E_3 \cup E_4) = P(E_1) + P(E_2) + P(E_3) + P(E_4) = \frac{4\binom{39}{13}}{\binom{52}{13}}.$$

However, the events E_1, E_2, E_3, E_4 are clearly **not** mutually exclusive because a hand can be void in several suits. To correct this we need to use the inclusion-exclusion principle. We have

$$P(E_1E_2) = P(E_1E_3) = P(E_1E_4) = P(E_2E_3) = P(E_2E_4) = P(E_3E_4) = \frac{\binom{26}{13}}{\binom{52}{13}},$$

$$P(E_1E_2E_3) = P(E_1E_2E_4) = P(E_1E_3E_4) = P(E_2E_3E_4) = \frac{\binom{13}{13}}{\binom{52}{13}} = \frac{1}{\binom{52}{13}}$$

and

$$P(E_1E_2E_3E_4) = 0.$$

By the inclusion-exclusion principle we have

$$P(E_1 \cup E_2 \cup E_3 \cup E_4) = 4 \cdot \frac{\binom{39}{13}}{\binom{52}{13}} - 6\frac{\binom{26}{13}}{\binom{52}{13}} + 4\frac{1}{\binom{52}{13}} =$$

$$= \frac{4\binom{39}{13} - 6\binom{26}{13} + 4}{\binom{52}{13}} = \frac{1621364909}{31750677980} \approx 0.0511$$

Exercise 6 (Chapter 2, theoretical exercises, #19)**.** Let us call the experiment in the exercise *Experiment 1*. *Experiment 2* is like *Experiment 1* except that we draw exactly k balls. This means that even if we have r red balls, but we do not have yet k balls in total we keep drawing more balls. Also if we have drawn k balls, we stop, even if we do not have yet r red balls. Let E be the event in Experiment 1 that a total of k balls were drawn. Let F be the the even in Experiment 2 that exactly r red balls were drawn and that the k-th ball was red. We have that E in Experiment 1 occurs if and only if F occurs in Experiment 2. Therefore

$$P(E) = P(F).$$

So we might as well deal with Experiment 2. Note that the order does matter in this problem. There are

$$n + m \cdot (n + m - 1) \cdot (n + m - k + 1) = \frac{(n+m)!}{(n+m-k)!}$$

ways of choosing k balls consecutively, so this is the size of the sample space. We now count $|F|$. There are $\binom{k-1}{r-1}$ to choose the positions for red and blue balls in the k drawings. For the red balls there are $\frac{n!}{(n-r)!}$ ways to choose and for the blue balls there are $\frac{m!}{(m-k+r)!}$ ways to choose. The answer is

$$\frac{\binom{k-1}{r-1} n! m! (n+m-k)!}{(n-r)!(m-k+r)!(n+m)!}.$$

Alternatively, we could first choose $k-1$ balls where we disregard the order of which $r-1$ have to be red, and then choose one more ball which have to be red. Viewed like this, the sample space has size $\binom{n+m}{k-1}(n+m-k+1)$. There are $\binom{n}{r-1}\binom{m}{k-r}$ ways of choosing the $r-1$ red balls and $k-r$ blue balls from the $k-1$ balls. After that, there are $n-r+1$ ways of choosing the last ball red. So the answer is

$$\frac{\binom{n}{r-1}\binom{m}{k-r}(n-r+1)}{\binom{n+m}{k-1}(n+m-k+1)}.$$

(Both answers are the same if worked out.)

Exercise 7. Suppose that A takes spinner a, and B takes spinner c. The sample space is

$$\{(i,j) \mid i \in \{1,5,9\}, j \in \{2,6,7\}\}.$$

Suppose that E is the event that B wins. Then

$$E = \{(1,2),(1,6),(1,7),(5,6),(5,7)\}.$$

So $P(E) = \frac{5}{9} > \frac{1}{2}$.

Similarly, A takes spinner b and B takes spinner a then the event that B wins is

$$\{(3,9),(4,9),(8,9),(3,5),(4,5)\}.$$

So again the probability that B wins is $\frac{5}{9}$.

If A takes spinner C then B should take spinner b. The event that B wins is

$$\{(2,3),(2,4),(2,8),(6,8),(7,8)\}$$

and the probability that B wins is $\frac{5}{9}$.

So with the right choice of spinner, B wins with probability $\frac{5}{9} > \frac{1}{2}$. So we'd rather be player B.