HOMEWORK 5, MATH 425, SECTION 3

HARM DERKSEN

Exercise 1 (Ch 4, problems, 43). A majority decoding algorithm decodes the digit correctly if the majority of the 5 transmitted digits is received correctly. Let X be the number of errors in the transmission of the 5 digits. Assuming that the errors in each of the 5 positions are independent of each other, we get that X is a binomial random variable with $n = 5$ and $p = 0.2$. The probability that we have at least 3 errors is

$$P(X \geq 3) = \binom{5}{3}(0.2)^3(0.8)^2 + \binom{5}{4}(0.2)^4(0.8) + \binom{5}{5}(0.2)^5 = 0.05792$$

Exercise 2 (Ch 4, problems, 46). Let X be the number of jury members who vote "guilty". If the defendent is guilty then X is a binomial random variable with $n = 12$ and $p = 0.8$. In this case the probability of a "guilty" verdict is

$$P(X \geq 9) = \binom{12}{9}(0.8)^9(0.2)^3+\binom{12}{10}(0.8)^{10}(0.2)^2+\binom{12}{11}(0.8)^{11}(0.2)+\binom{12}{12}(0.8)^{12} \approx 0.79457$$

and the probability of a "not guilty" verdict is

$$P(X \leq 3) = \binom{12}{0}(0.2)^{12}+\binom{12}{1}(0.2)^{11}(0.8)+\binom{12}{2}(0.2)^{10}(0.8)^2+\binom{12}{3}(0.2)^9(0.8)^3 \approx 0.00006.$$

If the defendent is innocent then X is binomial with $n = 12$ and $p = 0.1$. In this case the probability of a "guilty" verdict is

$$P(X \geq 9) = \binom{12}{9}(0.1)^9(0.9)^3+\binom{12}{10}(0.1)^{10}(0.9)^2+\binom{12}{11}(0.1)^{11}(0.9)+\binom{12}{12}(0.1)^{12} \approx 0.17 \cdot 10^{-6}.$$

and the probability of a "not guilty" verdict is

$$P(X \leq 3) = \binom{12}{0}(0.9)^{12}+\binom{12}{1}(0.9)^{11}(0.1)+\binom{12}{2}(0.9)^{10}(0.1)^2+\binom{12}{3}(0.9)^9(0.1)^3 \approx 0.97436.$$

Let C be the event of a correct decision, G be the event that the defendent is guilty and V the event of a "guilty" verdict. We have that $P(G) = .65$ and $P(G^c) = 1 - .65 = .35$. The probability of a correct decision is:

$$P(C) = P(C|G)P(G) + P(C|G^c)P(G^c) = P(V|G)P(G) + P(V^c|G^c)P(G^c) =$$

$$= P(X \geq 9|G)P(G)+P(X \leq 3|G^c)P(G^c) = (0.79457)(0.65)+(0.97436)(0.35) = 0.85750$$

The probability of a guilty verdict is

$$P(V) = P(V|G)P(G)+P(V|G^c)P(G^c) = (0.79457)(0.65)+(0.17 \cdot 10^{-6})(0.35) = 0.51647.$$

Exercise 3 (Ch 4, problems, 51)**.** Let X be the number of typographical errors. We assume that X is a Poisson random variable with parameter λ. We have $\lambda = EX = 0.2$.

(a)

$$P(X = 0) = e^{-\lambda} = e^{-0.2} \approx 0.8187.$$

(b)

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-0.2} - 0.2e^{-0.2} = 1 - (1.2)e^{-0.2} \approx 0.0175.$$

Exercise 4 (Ch 4, problems, 52)**.** Let X be the number of crashes in the next month. We get $\lambda = EX = 3.5$.
(**b**)

$$P(X \leq 1) = e^{-3.5} + (3.5)\, e^{-3.5} = 4.5\, e^{-3.5} \approx 0.1359.$$

(**a**)

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - 0.1359 = 0.8641.$$

Exercise 5 (Ch 4, problems, 75)**.** Let X_n be the number of correct numbers if the player chooses n numbers. Now X_n is a hypergeometric random variable with parameters $N = 80, m = 20$ and n.

(a) Now $n = 2$.

$$P(\text{win}) = P(X_2 = 2) = P_{2,2} = \frac{\binom{20}{2} \cdot \binom{60}{0}}{\binom{80}{2}} = \frac{19}{316}.$$

If the play wins x dollars if the two numbers are among the 20, then the expected gain is

$$\tfrac{19}{316}x - 1.$$

the payoff is fair of this expected gain is exactly 0. In that case we have

$$x = \frac{316}{19} \approx 16.632.$$

(b) We have

$$P_{n,k} = P(X_n = k) = \frac{\binom{20}{k} \cdot \binom{60}{n-k}}{\binom{80}{n}}.$$

(c) We have $n = 10$.

$P_{10,0}$	0.04579
$P_{10,1}$	0.17957
$P_{10,2}$	0.29526
$P_{10,3}$	0.26740
$P_{10,4}$	0.14732
$P_{10,5}$	0.05143
$P_{10,6}$	0.011479
$P_{10,7}$	0.0016111
$P_{10,8}$	0.00013542
$P_{10,9}$	0.000006121
$P_{10,10}$	0.0000001122

The expected payoff is

$$(0.04579+0.17957+0.29526+0.26740+0.14732)(-1)+(0.05143)(1)+(0.011479)(17)+$$
$$+(0.0016111)(179)+(0.00013542)(1299)+(0.000006121)(2599)+(0.0000001122)(24999) \approx -0.2057$$

Exercise 6 (Ch 4, theretical exercises, 19)**.** Assume that $n \geq 1$. We get, using the substitution $i = j + 1$, that

$$E(X^n) = \sum_{i=0}^{\infty} \frac{i^n \lambda^i}{i!} e^{-\lambda} = \sum_{i=1}^{\infty} \frac{i^n \lambda^i}{i!} e^{-\lambda} =$$
$$= \sum_{i=1}^{\infty} \frac{i^{n-1} \lambda^i}{(i-1)!} e^{-\lambda} = \sum_{j=0}^{\infty} \frac{(j+1)^{n-1} \lambda^{j+1}}{j!} e^{-\lambda} =$$
$$= \lambda \sum_{j=0}^{\infty} \frac{(j+1)^{n-1} \lambda^j}{j!} e^{-\lambda} = \lambda E((X+1)^{n-1}).$$

Using this we get

$$E(X^2) = \lambda E(X+1) = \lambda(\lambda+1) = \lambda^2 + \lambda$$

and

$$E(X^3) = \lambda E((X+1)^2) = \lambda(E(X^2) + 2E(X) + 1) = \lambda(\lambda^2 + \lambda + 2\lambda + 1) = \lambda^3 + 3\lambda^2 + \lambda.$$