HOMEWORK 9, MATH 425, SECTION 3

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Exercise 1 (Ch 6, problems 33). Let X be Jack's score, and Y be Jill's score.

(a) Put $W = X - Y$. then W is approximated by a normal distribution with $\mu_W = \mu_X - \mu_Y = 160 - 170 = -10$ and $\sigma_W = \sqrt{\sigma_X^2 + \sigma_Y^2} = \sqrt{15^2 + 20^2} = 25$.

$$P(W > 0.5) = P(Z > \frac{0.5 - (-10)}{25}) = 1 - \Phi(0.42) = 1 - .6628 = .3372.$$

(b) Put $W = X + Y$, then W is approximated by a normal with $\mu_Z = \mu_X + \mu_Y = 160 + 170 = 330$ and $\sigma_W = \sqrt{\sigma_X^2 + \sigma_Y^2} = \sqrt{15^2 + 20^2} = 25$.

$$P(W > 350) = P(Z > \frac{350.5 - 330}{25}) = 1 - \Phi(0.82) = 1 - .7939 = 0.2061.$$

Exercise 2 (Ch 6, problems 40). Let us first compute $p(i,j) = P(X = i, Y = j)$:

$p(i,j)$	$p(i,1)$	$p(i,2)$	$p(i,3)$	$p(i,4)$	$p(i,5)$	$p(i,6)$	$p_X(i)$
$p(1,j)$	$\frac{1}{36}$	0	0	0	0	0	$\frac{1}{36}$
$p(2,j)$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	$\frac{3}{36}$
$p(3,j)$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	$\frac{5}{36}$
$p(4,j)$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	$\frac{7}{36}$
$p(5,j)$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	$\frac{9}{36}$
$p(6,j)$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{11}{36}$
$p_Y(j)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$	

From this we compute $p_{Y|X}(j|i) = P(X = i, Y = j)/P(Y = j) = p(i,j)/p_Y(j)$:

$p_{Y\|X}(j\|i)$	$p_{Y\|X}(1\|i)$	$p_{Y\|X}(2\|i)$	$p_{Y\|X}(3\|i)$	$p_{Y\|X}(4\|i)$	$p_{Y\|X}(5\|i)$	$p_{Y\|X}(6\|i)$
$p_{Y\|X}(j\|1)$	1	0	0	0	0	0
$p_{Y\|X}(j\|2)$	$\frac{2}{3}$	$\frac{1}{3}$	0	0	0	0
$p_{Y\|X}(j\|3)$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	0	0
$p_{Y\|X}(j\|4)$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0	0
$p_{Y\|X}(j\|5)$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	0
$p_{Y\|X}(j\|6)$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{1}{11}$

The rows of the table above give the conditional mass functions. X and Y are not independent because $0 = p(1,2) \neq p_X(1)p_Y(2) = \frac{1}{36} \cdot \frac{9}{36}$.

Exercise 3 (CH 6, problems, 44). **(a)** We have

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} xe^{-x(y+1)}\,dy = -e^{-x(y+1)}\Big]_0^{\infty} = e^{-x}$$

if $x > 0$ and $f_X(x) = 0$ if $x \leq 0$.

We also have

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} xe^{-x(y+1)}\,dx = \frac{-xe^{-x(y+1)}}{y+1}\Big]_0^{\infty} + \int_0^{\infty} \frac{e^{-x(y+1)}}{y+1}\,dx =$$

$$= \frac{-e^{-x(y+1)}}{(y+1)^2}\Big]_0^{\infty} = \frac{1}{(y+1)^2}.$$

for $y \geq 0$ (and $f_Y(y) = 0$ for $y < 0$).

The density function for $Y|X = x$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = (y+1)^2 xe^{-x(y+1)}$$

for $x > 0$ (and $f(X|Y)(x,y) = 0$ otherwise). and

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xe^{-x(y+1)}}{e^{-x}} = xe^{-xy}.$$

if $y > 0$ and $f_{Y|X}(y|x) = 0$ otherwise.

(b)

$$P(Z \leq a) = P(XY \leq a) = \int_0^{\infty}\int_0^{a/x} xe^{-x(y+1)}\,dy\,dx = \int_0^{\infty} -e^{-x(y+1)}\Big]_0^{a/x}\,dx =$$

$$= \int_0^{\infty} e^{-x} - e^{-x-a}\,dx = 1 - e^{-a}.$$

Differentiation yields

$$f_Z(a) = e^{-a}$$

for $a \geq 0$ (and $f_Z(a) = 0$ for $a < 0$).

Exercise 4 (CH 7, problems, 8). We have

$$EX_i = P(X_i = 1) = (1-p)^{i-1},$$

hence

$$E(X) = E(\sum_{i=1}^{N} X_i) = \sum_{i=1}^{N} EX_i = \sum_{i=1}^{N}(1-p)^{i-1} = \sum_{i=0}^{N-1}(1-p)^i = \frac{1-(1-p)^N}{1-(1-p)} = \frac{1-(1-p)^N}{p}$$

where X is the number of occupied tables.

Exercise 5 (CH 7, problems, 12). **(a)** Let Y be the number of men with at least a female neighbor. Define $X_i = 1$ if the i-th man has no female neighbors. We have $EX_1 = P(X_1 = 1)$ is exactly the probability that the first two people in line are both men. The probability of that is

$$\frac{\binom{n}{2}}{\binom{2n}{2}} = \frac{(n)(n-1)}{2n(2n-1)} = \frac{n-1}{4n-2}.$$

Similarly $EX_n = (n-1)/(4n-2)$. If $1 < i < n$ then $EX_i = P(X_i = 1)$ is the probability that both neighbors of the i-th man are male. We count the number of sequences with n M's and n F's such that the i-th M has an M on each side. This means that the $i-1$-th and i-th and $i+1$-th man are next to each other. Let us replace these three by one man. Then we get a sequence of length $2n-2$ with $n-2$ men and n women. In fact, this way we get every such sequence exactly once. So the number of sequences is $\binom{2n-2}{n-2}$. The probability that man i has two male neighbors is

$$\frac{\binom{2n-2}{n-2}}{\binom{2n}{n}} = \frac{(2n-2)!n!n!}{(2n)!n!(n-2)!} = \frac{n-1}{4n-2}.$$

Therefore $EX_i = (n-1)/(4n-2)$ for all i. We conclude

$$EY = E(n - \sum_{i=1}^{n} X_i) = n - \sum_{i=1}^{n} EX_i = n - n\frac{n-1}{4n-2} = \frac{3n^2-n}{4n-2}.$$

(b) Label the men by $1, 2, \ldots, n$. We have $EX_i = P(X_i = 1)$ is the probability that both neighbors of man i are male. The probability of this is

$$\frac{\binom{n-1}{2}}{\binom{2n-1}{2}} = \frac{(n-1)(n-2)}{(2n-1)(2n-2)} = \frac{n-2}{2n-2}.$$

We conclude

$$EY = E(n - \sum_{i=1}^{n} X_i) = n - \sum_{i=1}^{n} EX_i = n - n\frac{n-2}{4n-2} = \frac{3n^2}{4n-2}.$$

Exercise 6 (Ch7, theoretical exercises, 1).
$E((X-a)^2) = E(X^2 - 2aX + a^2) = E(X^2) - 2aEX + a^2 = (a - EX)^2 + E(X^2) - (EX)^2$ which is clearly minimal if $a = EX$.