Problem 1. Chapter 2, Exercise 7. For each pair of axioms of incidence geometry, invent an interpretation in which those two axioms are satisfied, but the third axiom is not.

I-1, I-2 satisfied, I-3 not: Take a geometry for which there are no lines, and no points. Since there are no points, I-1 is satisfied, because one cannot find 2 distinct points. Similarly I-2 is satisfied, because there does not exist any line. I-3 is not satisfied because I-3 tells us that there exist at least 3 points, and there are none in our model.

Another example: The geometry with 1 point and no lines.

Yet another example: Take the geometry with \( n \) points, where \( n \geq 2 \), and one line \( \ell \) which is incident with all points. Through every two distinct points there is a unique line, namely \( \ell \). So I-1 is satisfied. Every line contains at least 2 points, so I-2 is satisfied. But I-3 is not satisfied because every 3 points lie on the line \( \ell \), so one cannot find 3 points which are not collinear.

I-1, I-3 satisfied, but not I-2: Let \( A, B, C \) be the points, and the lines are \{\( A, B \), \( A, C \), \( B, C \), \( A \}\}. Clearly \( A, B, C \) do not lie on a line so I-3 is satisfied. The unique line through \( A, B \) is \( \{A, B\} \) and similarly there are unique lines through \( B, C \) and through \( A \) and \( C \). So I-1 is satisfied. But I-2 is not satisfied because the line \( \{A\} \) does not contain 2 distinct points.

I-2, I-3 satisfied, but not I-1: Take the geometry with \( 3 \) points \( A, B, C \) and no lines. Then I-2 and I-3 are clearly satisfied, but I-1 is not because there is no line through \( A \) and \( B \).

Problem 2. Chapter 2, Exercise 8. Show that the interpretations of Examples 3 and 4 of this chapter are models of incidence geometry and that the Euclidean and hyperbolic parallel properties, respectively, hold for them.

Both in Example 3 and Example 4, all lines have exactly 2 points on them. In particular Axiom I-2 is satisfied in both examples.

Both in Example 3 and Example 4 we have the following property. If \( P \) and \( Q \) are distinct points then \( \{P, Q\} \) is a line. Since every line has 2 points, the only line containing \( P \) and \( Q \) is \( \{P, Q\} \). So \( \{P, Q\} \) is the unique line through \( P \) and \( Q \) and I-1 is satisfied.

Both in Example 3 and Example 4, the points \( A, B, C \) are distinct and not collinear, so I-3 is satisfied.

In example 3, if we take the line \( \{A, B\} \) and the points \( C \) which does not lie on \( \{A, B\} \), then there is a unique line through \( C \) parallel to \( \{A, B\} \) namely \( \{C, D\} \) (the other lines through \( C \) are \( \{A, C\}, \{B, C\} \) but they intersect \( \{A, B\} \)). By relabeling we see that for every line \( \ell \) and
every point \( P \), not on \( \ell \) there exists a unique line \( m \) which is parallel to \( \ell \) and goes through \( P \). The Euclidean parallel property is satisfied.

In example 4, if we take the line \( \{A, B\} \) and the point \( C \), then there exist two lines through \( C \) which are parallel to \( \{A, B\} \), namely \( \{C, D\} \) and \( \{C, E\} \). By symmetry again, we see that for every line \( \ell \) and every point \( P \) not on \( \ell \) there exist two lines through \( P \) which are parallel to \( \ell \).

**Problem 3.** Chapter 2, Exercise 14.

(a) Let \( S \) be the following statement in the language of incidence geometry: if \( \ell \) and \( m \) are any two distinct lines, then there exists a point \( P \) that does not lie on either \( \ell \) or \( m \). Show that \( S \) is not a theorem in incidence geometry, i.e., cannot be proved from the axioms of incidence geometry.

(b) Show, however, that statement \( S \) holds in every projective plane. Hence \( \sim S \) cannot be proved from the axioms of incidence geometry either, so \( S \) is independent of those axioms.

(c) Use statement \( S \) to prove that in a finite projective plane, all the lines have the same number of points lying on them.

(d) Prove that in a finite affine plane, all the lines have the same number of points lying on them.

(a) There exist a model for which \( S \) is not true, namely example 1. The points are \( A, B, C \) and the lines are \( \{A, B\}, \{A, C\} \) and \( \{B, C\} \). There is no point that does not lie on one of the lines \( \{A, B\}, \{A, C\} \).

(b) Suppose that \( \ell \) and \( m \) are two distinct lines. They intersect at a unique point \( P \) because the plane is projective. Since \( \ell \) and \( m \) have at least 3 points, we can find a point \( Q \) on \( \ell \) and a point \( R \) on \( m \) such that \( Q \) and \( R \) are not equal to \( P \). If \( Q \) lies on \( m \) then \( Q \) lies on \( \ell \) and \( m \), so \( P = Q \) because \( P \) is the unique intersection point of \( \ell \) and \( m \). Contradiction. So \( Q \) does not lie on \( m \) and \( R \) does not lie on \( \ell \). In particular, \( Q \) is not equal to \( R \). The line \( \overline{QR} \) contains at least one more point, say \( S \). Clearly \( \overline{QR} \) is not equal to \( \ell \) because \( R \) does not lie on \( \ell \). The unique intersection point of \( \overline{QR} \) and \( \ell \) is \( Q \). Since \( S \) lies on \( \overline{QR} \) and \( S \) is not equal to \( Q \), we have that \( S \) does not lie on \( \ell \). Similarly, \( S \) does not lie on \( m \).

(c) Suppose that \( \ell \) and \( m \) are distinct lines, and \( P \) is a point not on \( \ell \) or \( m \) (which exists by (b)). If \( Q \) is a point on \( \ell \) then consider the line \( \overline{PQ} \). The line \( \overline{PQ} \) is not equal to \( m \) because \( P \) does not lie on \( m \). So these two lines intersect at a unique point, call it \( Q' \). So to every point \( Q \) on \( \ell \) we can associate a point \( Q' \) on \( m \). Suppose that \( Q, R \) are distinct points on \( \ell \) and \( Q' = R' \). Then \( Q' = R' \) lies on \( \overline{PQ} \) and on \( \overline{PR} \). Now \( Q' \) is not equal to \( P \) because \( P \) does not lie on \( m \). By the uniqueness property in Axiom I-1, \( \overline{PQ} = \overline{PR} \). This means that \( R \) and \( Q \) both lie on \( \ell \) and on \( \overline{PQ} \). Since \( \ell \neq \overline{PQ} \) (\( P \) does not lie on \( \ell \)) we must have \( Q = R \). In other words, distinct points on \( \ell \) give distinct points on \( m \). This shows that \( m \) has at least as many points as \( \ell \). By reversing the roles of \( \ell \) and \( m \) we also see that \( \ell \) has at least as many points as \( m \). So we conclude that \( \ell \) and \( m \) have the same number of points.

(d) Suppose that \( \ell \) and \( m \) are lines in the affine plane. In the projective completion, \( \ell \) and \( m \) each have an additional point at infinity. In the projective plane, \( \ell \) and \( m \) have the same number of points. Therefore, \( \ell \) and \( m \) also have the same number in the affine plane, because
we just remove one point at infinity for each if we go from the projective plane to the affine plane.

**Problem 4.** Chapter 2, Exercise 15.

(a) Four distinct points, no three of which are collinear, are said to form a *quadrangle*. Let $\mathcal{P}$ be a model of incidence geometry for which every line has at least 3 distinct points lying on it. Show that a quadrangle exists in $\mathcal{P}$.

(b) Now suppose $\mathcal{P}$ is a projective plane. Four distinct lines, no three of which are concurrent, are said to form a *quadrilateral*. Use the principal of duality to prove that a quadrilateral exists in $\mathcal{P}$.

(c) Give an example of a statement that holds in all affine planes but whose dual never holds. Thus the principle of duality is not valid for affine planes.

(a) By axiom I-1 there exist points $A, B, C$ which are not collinear. Now $\overrightarrow{AB}$ contains at least one other point not equal to $A$ or $B$, say $D$. and $\overrightarrow{AC}$ contains at least one other point not equal to $A$ or $C$, say $E$. Now $B, C, D, E$ form a quadrangle: Suppose that $B, C, D$ lie on a line. The unique line through $B$ and $D$ is $\ell$ (Axiom I-1). And $C$ lies on the unique line through $B$ and $D$. So $A, B, C$ all lie on $\ell$. Contradiction. So $B, C, D$ are not on a line. Similarly, $B, C, E$ are not collinear, $B, D, E$ are not collinear, and $C, D, E$ are not collinear.

(b) The dual $\mathcal{P}^*$ is also a projective plane, so there exist 4 points $A, B, C, D$ such that no 3 are on a line. The points $A, B, C, D$ in the dual plane $\mathcal{P}^*$ correspond to lines $a, b, c, d$ in the plane $\mathcal{P}$. Three of the lines $a, b, c, d$ are concurrent if and only if the corresponding points are collinear. Since no 3 of the points $A, B, C, D$ are collinear, no three lines from $a, b, c, d$ are concurrent.

(c) “For every two distinct points $P$ and $Q$ there exists a line through $P$ and $Q$.” This follows immediately from axiom I-1. The dual statement is: “Every two distinct lines have at least 1 point in common.” This statement is never true in an affine plane. Indeed, let $\ell$ be a line, and $A$ be a point not on $\ell$ (which exists by Proposition 2.3). There exists a line $m$ through $A$ which is parallel to $\ell$. But then $\ell$ and $m$ don’t intersect.

**Problem 5.** Chapter 2, Major Exercises 3.

Let $\mathcal{P}$ be a finite projective plane so that, according to Exercise 14(c), all lines in $\mathcal{P}$ have the same number of points lying on them; call this number $n + 1$, with $n \geq 2$. Show the following:

(a) Each point in $\mathcal{P}$ has $n + 1$ lines passing through it.

(b) The total number of points in $\mathcal{P}$ is $n^2 + n + 1$.

(c) The total number of lines in $\mathcal{P}$ is $n^2 + n + 1$.

(a) Let $P$ be a point, and choose a line $\ell$ which does not go through $P$ (Proposition 2.4). Let $Q_1, Q_2, \ldots, Q_{n+1}$ be the points of $\ell$. If $\overrightarrow{PQ_i} = \overrightarrow{PQ_j}$ and $i \neq j$, then $P, Q_i, Q_j$ are collinear, and $P$ lies on line $Q_iQ_j = \ell$. Contradiction. The lines $\overrightarrow{PQ_1}, \overrightarrow{PQ_2}, \ldots, \overrightarrow{PQ_{n+1}}$ are distinct. So we have $n + 1$ lines through $P$. But perhaps there are more. Suppose that $m$ is any line through $P$. Then $m$ must intersect with $\ell$, say they intersect at $Q_i$. Then $P$ and $Q_i$ both lie on $m$ and $\overrightarrow{PQ_i}$, so $m = \overrightarrow{PQ_i}$. This shows that there are exactly $n + 1$ lines through $P$. 
(b) Suppose that $R$ is a point not equal to $P$. Then $\overrightarrow{PR}$ is equal to $\overrightarrow{PQ_i}$ for some $i$ and $R$ lies on $PQ_i$. The $i$ is unique because the lines $PQ_1, PQ_2, \ldots, PQ_{n+1}$ only intersect at $P$. Each of the lines $PQ_i$ has $n + 1$ points, and $n$ points not equal to $P$. So in total there are $n(n + 1)$ points which are not equal to $P$. So the total number of points is $n(n + 1) + 1 = n^2 + n + 1$.

(e) By dualizing part (a), we see that every line in the dual plane $P^*$ has exactly $n + 1$ points. By part (b), $P^*$ has $n^2 + n + 1$ points. By dualizing, we see that $P$ has $n^2 + n + 1$ lines.