MATH 513: LINEAR ALGEBRA
ASSIGNMENT 5

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The **Challenging Problems** are due on Friday, October 12 at noon in class. You do **not** have to hand in the routine problems. On a quiz on Monday, October 15, similar problems may appear. It is optional to hand in the **Very Challenging Problems** (but the same deadline applies). These problems will be very hard. You can earn extra credit with the very challenging problems (but they will be graded more strictly).

Reading

Read up to section 10 if you haven't done so already. We may do some review of Chapter 2.

Routine Problems

1. Do Section 9, page 68, problem 1.
2. Do Section 9, page 68, problem 3.
3. Do Section 9, page 68, problem 4.

Challenging Problems

1. Do Section 9, page 68/69, problem 5.
2. Do Section 9, page 69, problem 6.

3. Study Example E on page 67/68. Let us consider the case $m = 1$. The set of solutions of the differential equation

$$y'' + y = 0 \tag{1}$$

is a subspace of $\mathcal{F}(\mathbb{R})$ (the realvalued functions on $\mathbb{R}$). We have two linearly independent solutions, namely $y_1 = \cos(x)$ and $y_2 = \sin(x)$. A result from analysis says that for given $\alpha, \beta \in \mathbb{R}$ there is a *unique* solution of (1) satisfying $y(0) = \alpha$ and $y'(0) = \beta$. (The only things you need to know about cos and sin in this problem are: $\cos(0) = 1$, $\sin(0) = 0$, $\cos' = -\sin$ and $\sin' = \cos$.)
   (a) Prove that if y is a solution of (1) then $y = \alpha \cos(x) + \beta \sin(x)$ with $\alpha = y(0)$ and $\beta = y'(0)$. In particular, $\{\cos(x), \sin(x)\}$ is a basis of the solution space.
   (b) Check that for any $\alpha \in \mathbb{R}$, $\cos(x + \alpha)$ and $\sin(x + \alpha)$ are also solutions of (1). Then prove the following formulas

$$\cos(x + \alpha) = \cos(\alpha)\cos(x) - \sin(\alpha)\sin(x)$$

   and

$$\sin(x + \alpha) = \sin(\alpha)\cos(x) + \cos(\alpha)\sin(x)$$

   (Note: Although the roles of x and α in the formula are symmetric, their roles in the proof are not. In the proof of the formula we consider x to be a variable and $\alpha \in \mathbb{R}$ to be a constant.)

Very Challenging Problems

1. Let $p_1(x), p_2(x), \ldots, p_k(x)$ be polynomial functions on $\mathbb{R}$ of degree $\leq n$. Show that if

$$(k - 1)(d + 1) > n$$

then there exists polynomial functions $a_1(x), a_2(x), \ldots, a_k(x)$ of degree $\leq d$ such that

$$a_1(x)p_1(x) + a_2(x)p_2(x) + \cdots + a_k(x)p_k(x) = 0.$$