# MATH 513: LINEAR ALGEBRA
# "ASSIGNMENT" 8

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**MIDTERM** is on Friday, November 2, on Chapters 1,2,3. **NO** problems need to be handed in and there will be **NO** quiz on Monday November 5.

## Reading

Read Section 14 and 15. Of course, also study for the exam.

## Routine Problems

1. Do Section 12, page 99, problem 7,8.
2. Do Section 13, page 107, problem 1,2,3,7,8,9.

## 1. Practice Exam

Here is a practice exam (an exam of a previous year). The midterm exam may vary slightly in difficulty, length and emphasis.

1. (5 points) Let $\{v_1, \dots, v_n\}$ be a finite set of vectors in a vector space $V$ over a field $F$. Define what it means for $\{v_1, v_2, \dots, v_n\}$ to be linearly *dependent*.
2. (10 points) Let $V$ be a vector space over a field $F$. Define what it means for the dimnesion of $V$ to be equal to some integer $n > 0$, and state (as percisely as possible) the theorem which ensures that the dimension of $V$ is well-defined.
3. (15 points) Consider the following system of equations, where the variables $x_1, x_2, x_3, x_4$ and $x_5$ all take values in $\mathbb{F}_2$.

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & & & + & x_4 & & & = & 1 \\ & & x_2 & + & x_3 & & & & & = & 1 \\ x_1 & + & x_2 & & & & & + & x_5 & = & 1 \\ & & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 1 \end{array}$$

   Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ to these equations, if any. You do not need to explain your answer, but show all your work.
4. Are the following statements true or false? If they are false, show it (by giving a counterexample).
   (a) (10 points) Let $V$ be a vector space over a field $V$ with $\dim V = n > 2$. If $v_1$ and $v_2$ are nonzero vectors in $V$ then $V$ has some basis

$\{v_1, v_2, v_3, \ldots, v_n\}$. (In other words, the set $\{v_1, v_2\}$ may be extended to a basis for $V$).

(b) (10 points) The vectors $(2,0,1,2), (5,0,-1,7), (-13,0,1,3)$ and $(-6,0,2,-1)$ are a basis for $\mathbb{R}^4$.

(c) (10 points) Let $F$ be a field and for $\alpha_{i,j} \in F$ and $\beta_i \in F$ let $S$ be the set of all $(x_1, x_2, x_3)$ such that

$$\begin{aligned} \alpha_{1,1}x_1 + \alpha_{1,2}x_2 + \alpha_{1,3}x_3 &= \beta_1 \\ \alpha_{2,1}x_1 + \alpha_{2,2}x_2 + \alpha_{2,3}x_3 &= \beta_2 \end{aligned}$$

Then $S$ is a subspace of $F^3$.

(d) (10 points) Let $V$ be a nonzero vector space over a field $F$ such that $V = S(v_1, \ldots, v_n)$ for some positive integer $n$. Then for $m > n$, and vectors $w_1, \ldots, w_m \in V$, $\{w_1, \ldots, w_m\}$ are linearly dependent.

5. (10 points) This is a proof question. Let $T$ be a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^7$ and let $\mathbf{A}$ be the matrix of $T$ with respect to the standard basis of $\mathbb{R}^4$ and $\mathbb{R}^7$. Prove that if the nullspace of $T$ has dimension 1, then the row rank of $\mathbf{A}$ is 3.
6. (20 points) This is a proof question. Let $T$ be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ and let $\mathbf{A}$ be the matrix of $T$ with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^4$. Prove that the following conditions are equivalent.
   (a) There exists some $S \in L(\mathbb{R}^4, \mathbb{R}^3)$ such that $ST$ is the identity in $L(\mathbb{R}^3, \mathbb{R}^4)$;
   (b) The columns of $\mathbf{A}$ are linearly independent.

## Additional Practice questions for the Midterm Exam

1. Let $F$ be a field and $\alpha, \beta \in F$. Prove that if $\alpha\beta = 0$ then $\alpha = 0$ or $\beta = 0$.
2. Suppose that $v_1, v_2, \ldots, v_n$ are vectors in an $n$-dimensional vector space $V$. Assume that $v_1 \neq 0$ and $v_i \notin S(v_1, v_2, \ldots, v_{i-1})$ for $i = 2, 3, \ldots, n$. Prove that $\{v_1, v_2, \ldots, v_n\}$ is a basis of $V$.
3. Prove that $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$ for two $n \times m$ matrices $A$ and $B$.
4. Are the vectors $(1,2,0)$, $(0,1,2)$, $(2,0,1)$ in $\mathbb{F}_3^3$ linearly dependent?
5. Prove: If $T : V \to W$ is linear and injective, and $V$ and $W$ are finitely generated, then there exists a linear transformation $U : W \to V$ such that $UT = 1$.
6. True or False: Suppose $v, w, z$ are vectors in a vector space $V$. If $\{v, w\}$ are linearly dependent, and $\{w, z\}$ are linearly dependent, then $\{v, z\}$ are linearly dependent.