MATH 513: LINEAR ALGEBRA
ASSIGNMENT 9

HARM DERKSEN

The **Challenging Problems** are due on **Friday, November 16** at noon in class. You do **not** have to hand in the routine problems. On a quiz on Monday, November 19, similar problems may appear. It is optional to hand in the **Very Challenging Problems** (but the same deadline applies). These problems will be very hard. You can earn extra credit with the very challenging problems (but they will be graded more strictly).

I will travel abroad from Saturday November 10 until Thursday November 15. The lectures on Monday November 12 and November 14 will be held by Calin Chindris (same time, same place). There will be **no** office hours on Monday or Thursday. Students with questions can email me: `hderksen@umich.edu`. I intend to read my mail at least a few times while I am abroad.

Reading

Read Chapter 5 on determinants.

Routine Problems

1. Do Section 14, page 118, problem 1.
2. Do Section 14, page 119, problem 4.
3. Do Section 15, page 129, problem 1.
4. Do Section 15, page 129, problem 2.
5. Do Section 15, page 129, problem 7 (The definition of a group is on page 82).
6. Prove that $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$
7. Do Section 16, page 139, problem 1.

Challenging Problems

1. Do Section 15, page 130, problem 9.
2. Do Section 15, page 131, problem 12.
3. Do Section 15, page 131, problem 13.
4. Suppose that V is a vector space over $\mathbb{R}$.
   (a) Recall that a function $B(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called a *bilinear form* if

$$B(u+v,w) = B(u,w) + B(v,w)$$

$$B(u, v + w) = B(u, v) + B(u, w)$$
$$B(\alpha u, v) = \alpha B(u, v) = B(u, \alpha v)$$

for all $u, v \in V$ and all $\alpha \in \mathbb{R}$. If B_1, B_2 are bilinear forms, define their sum $B_1 + B_2$ by

$$(B_1 + B_2)(v, w) = B_1(v, w) + B_2(v, w)$$

for all $v, w \in V$. If B is a bilinear form and $\lambda \in \mathbb{R}$, define λB by

$$(\lambda B)(v, w) = \lambda(B(v, w)).$$

Let $\mathrm{Bil}(V)$ be the set of bilinear forms on V. Show that $\mathrm{Bil}(V)$ is a vector space.

(b) If B is a bilinear form on V, $v = \sum_{i=1}^{n} \xi_i v_i$ and $w = \sum_{i=j}^{n} \eta_j$. show that

$$B(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i,j} \xi_i \eta_j$$

where $\alpha_{i,j} = B(v_i, v_j)$. We call $\mathbf{A} = (\alpha_{i,j})$ the matrix of B with respect to the basis $v_1, v_2, \dots, v_n$. Also show that for any $n \times n$ matrix $\mathbf{A}$, one can define a bilinear form B whose matrix (w.r.t. the basis $\{v_1, \dots, v_n\}$) is $\mathbf{A}$. Give a basis of $\mathrm{Bil}(V)$. What is the dimension of $\mathrm{Bil}(V)$?

(c) The bilinear form is called symmetric if $B(v, w) = B(w, v)$ for all $v, w \in V$. Let $\mathrm{SBil}(V)$ be the set of symmetric bilinear forms on V. Show that $\mathrm{SBil}(V)$ is a subspace of $\mathrm{Bil}(V)$. What is the dimension of $\mathrm{SBil}(V)$?

(d) A symmetric bilinear form is an *inner product* if $B(v, v) \geq 0$ for all $v \in V$ and $B(v, v) = 0$ implies $v = 0$. Is the set of inner products a subspace of $\mathrm{Bil}(V)$?

5. Do Section 16, page 139, problem 3.

Very Challenging Problems

1. (Not very hard this time, I think) Suppose that a, b, c are vectors in a vector space V over $\mathbb{R}$ with an inner product satisfying $a + b + c = 0$. Assume that each of the pairs $\{a, b\}$, $\{b, c\}$, $\{a, c\}$ is linearly independent. Let α be the angle between b and c, β be the angle between a and c and γ be the angle between a and b ($\alpha, \beta, \gamma, \in [0, \pi]$, see definition 15.7). Prove the following, generalized sine rule:

$$\frac{\|a\|}{\sin \alpha} = \frac{\|b\|}{\sin \beta} = \frac{\|c\|}{\sin \gamma}.$$