CODING THEORY, MATH 567
MIDTERM TAKE HOME EXAM

DUE: MARCH 24

You should work on this midterm exam individually.

(1) You roll a die. If the number of eyes is not 6, then you roll it again. You keep rolling the die until you throw 6 or until you have thrown the die n times, whatever happens first. Give a formula (an expression involving n) for the total entropy of this procedure. What happens if $n \to \infty$?

(2) Suppose that $X = (X_1, X_2, \ldots, X_{120})$ is a random vector of length 120. For every i we have that $X_i = 0$ with probability 0.9 and $X_i = 1$ with probability 0.1. Moreover, $X_1, X_2, \ldots, X_{120}$ are independent random variables.

  (a) What is the *binary* entropy $H_2(X)$ of X?

  (b) We encode the outcome of X as follows. Suppose that X has outcome $a_1 a_2 \cdots a_{120}$. We cut this binary word in blocks of size n. Then we encode each block of n bits using a Huffman code $f : \{0,1\}^n \to \{0,1\}^\star$. Describe such a Huffman code explicitly for $n = 3$.

  (c) Show that for some n this encoding scheme will give an average length of less than 60 (so more than $2\times$ compression.) What is the smallest such n?

(3) Suppose that we have two discrete memoryless channels C_1 and C_2 with channel capacities $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. Let $\{x_1, x_2, \ldots, x_r\}$ be the input alphabet of C_1, $\{y_1, y_2, \ldots, y_s\}$ is the output alphabet of C_1 as well as the input alphabet of C_2 and $\{z_1, z_2, \ldots, z_t\}$ is the ouput alphabet of C_2. Since the output alphabet of C_1 is the same as the input alphabet of C_2, we can concatenate the two channels in a natural way to form a new channel with input alphabet $\{x_1, x_2, \ldots, x_r\}$ and output alphabet $\{z_1, z_2, \ldots, z_t\}$. This concatenation channel will be denoted by C_3.

  (a) If A_1 is the channel matrix of C_1 and A_2 is the channel matrix of C_2, what is the channel matrix of C_3?

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(b) Suppose we have some distribution of the input $\{x_1, x_2, \ldots, x_r\}$ of C_1. Let X be the corresponding random variable. This induces a probability distributions on $\{y_1, y_2, \ldots, y_s\}$ and on $\{z_1, z_2, \ldots, z_t\}$. The corresponding random variables will be denoted by Y and Z. Prove that $H(Z|Y) = H(Z|(X,Y))$. What does this mean in words?

(c) Prove that $I(X,Z) \leq I(X,Y)$, $I(X,Z) \leq I(Y,Z)$. Let $\mathcal{C}_3$ be the capacity of C_3. Show that $\mathcal{C}_3 \leq \mathcal{C}_1$, $\mathcal{C}_3 \leq \mathcal{C}_2$.

(4) Do §4.3, problem 24: Apply the $(u, u+v)$-construction to the $(8, 20, 3)$ code (you may just assume that this code exists) and another code (which one?) to get a $(16, 2560, 3)$ code.

(5) What bound for $\alpha_q(\delta)$ does the Singleton bound give? Also, use the Hamming bound to find an assymptotic upper bound for $\alpha_q(\delta)$.

(6) Do §4.5, problem 25.