CODING THEORY, MATH 567
PROBLEM SET 1
DUE: FRIDAY FEBRUARY 2

(1) **§1.1, Exercise 11:** Suppose we roll a fair die that has two faces numbered 1, two faces numbered 2, and two faces numbered 3. Then we toss a fair coin the number of times indicated by the number on the die. How much information do we get by this procedure?

(2) **§1.1, Exercise 12:** The accuracy of a certain radio station's weatherman at predicting rain is given by the following chart:

	actual rain	actual no rain
predicts rain	1/12	1/6
predicts no rain	1/12	2/3

For instance, 1/12 of the time the weatherman predicts rain when in fact it does rain. Notice that the weatherman is correct 3/4 of the time. Now an unemployed listener observes that he could be correct 5/6 of the time by simply always predicting no rain, and so he applies for the weatherman's job. However, the station manager declines to hire the listener. Why?

Hint: Think in terms of random variables and entropy. Let X be the random variable with range {actual rain, actual no rain}, Y be the weatherman's prediction and Z be the listener's prediction. What are $H(X), H(Y), H(Z), H(X,Y), H(X,Z)$? Compute $I(X,Y) := H(X) + H(Y) - H(X,Y)$. Here $I(X,Y)$ denotes the mutual information of X and Y, roughly the amount of information that the outcome of Y has about X. Also compute $I(X,Z)$. How do $I(X,Y)$ and $I(X,Z)$ compare?

(3) **§1.1, Exercise 15:** Let $\mathcal{S}_1 = (S_1, P_1)$ and $\mathcal{S}_2 = (S_2, P_2)$ be sources, with $S_1 = \{x_1, \ldots, x_n\}$, $P_1(x_i) = p_i$ and $S_2 = \{y_1, \ldots, y_m\}$, $P_2(y_i) = q_i$. Let $\lambda, \mu \geq 0$, $\lambda + \mu = 1$. Define the *mixed source* $\mathcal{S} = \lambda\mathcal{S}_1 + \mu\mathcal{S}_2$ to have alphabet $S_1 \cup S_2$ and probabilities $P(x_i) = \lambda_i p_i$, $P(y_i) = \mu q_i$.

(a) Calculate the entropy of $\mathcal{S}$.

Date: Winter 2007.

(b) Determine the value of λ that maximizes the entropy.

(4) **§1.1, Exercise 18:**

(a) A personal computer monitor is capable of displaying pictures made up of pixels at a resolution of 640 columns by 480 rows. If each pixel can be in any of 16 colors, estimate the amoount of information in a random picture.

(b) Estimate the information obtained from a random speech of 1,000 words, assumin a 10,000 word vocabulary.

(This shows that a picture is actually worth more than a thousand words!)

(5) **§1.2, Exercise 9:** Let X be a random variable, and let $Y = f(X)$. Prove that $H(Y) \leq H(X)$. Show that equality holds if and only if f is one-to-one (injective) on the set of all x such that $P(X = x) \neq 0$.

Note: If X is a random variable with range $\{x_1, x_2, ..., x_n\}$ and

$$f : \{x_1, x_2, \ldots, x_n\} \to \{y_1, y_2, \ldots, y_m\}$$

is a map, then $Y = f(X)$ is a random variable with range $\{y_1, y_2, \ldots, y_m\}$ such that

$$P(Y = y_i) = \sum_{j, f(x_j)=y_i} P(X = x_j).$$

for all i. (Whenever X has outcome x_j, then Y has outcome $f(x_j)$.)

(6) Do §1.2, Exercise 10: Let $P_1 = \{p_1, \ldots, p_n\}$ be a probability distribution, with $p_1 \geq p_2 \geq \cdots \geq p_n$. Suppose that $\varepsilon > 0$ has the property that $p_1 - \varepsilon \geq p_2 + \varepsilon$. Show that

$$H(p_1, \ldots, p_n) \leq H(p_1 - \varepsilon, p_2 + \varepsilon, p_3, \ldots, p_n).$$

Interpret this in words.

Hint: One can use the convexity of the entropy.