MATH 567, WINTER 2007
MIDTERM, DUE MONDAY, MARCH 19, 2007

(1) For each of the following channel matrices, determine (i) whether the channel is deterministic, (ii) whether the channel is lossless, (iii) whether the channel is rowsymmetric and (iv) whether the channel is column symmetric.
(a) \[
\begin{pmatrix}
\frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
(c) \[
\begin{pmatrix}
1 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]
(d) \[
\begin{pmatrix}
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4}
\end{pmatrix}
\]
(e) \[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
0 & 1
\end{pmatrix}
\]

(2) For each of the channels in problem (1), compute its capacity (say in base 2).

(3) Suppose that $X$, $Y$ and $Z$ are discrete random variables.
(a) Prove the inequality
\[H(X,Y|Z) \leq H(X|Z) + H(Y|Z) .\]
(b) Prove that
\[H(X,Y,Z) + H(Z) \leq H(X,Z) + H(Y,Z)\]
(all entropies are in the same base).

(4) Suppose that $C = (c_1, c_2, \ldots, c_m)$ is a maximal instantaneous $r$-ary encoding scheme. Prove that there exist probabilities $p_1, p_2, \ldots, p_m$ with $p_1 + p_2 + \cdots + p_m = 1$ such that $C$ is optimal for the probability distribution $P = (p_1, p_2, \ldots, p_m)$ (in the sense that it has minimal average length for this distribution).
(5) Consider the repetition code $C = \{000, 111\}$ and the binary symmetric channel with crossover probability $p = 1/3$. Assume that codeword 000 is sent with probability $3/4$ and 111 is sent with probability $1/4$.

(a) How will 110 be decoded if we use an ideal observer?

(b) What is the decision error probability $p_e$ if we use an ideal observer?

(c) How will 110 be decoded if we use maximal likelyhood decoding?

(d) What is the average maximum decision error $p_e^{\text{max}}$?

(e) What is the average decision error $p_e^{\text{av}}$?