Exercise 1. Make a table of all conjugacy classes of $S_6$. For each conjugacy class, compute its cardinality, and the cardinality of the centralizer of a representant. (Check: Do the cardinalities of the conjugacy classes add up to $6! = 720$?)

Solution.

<table>
<thead>
<tr>
<th>partition</th>
<th>example</th>
<th>card. of conj. class</th>
<th>card. of centralizer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1, 1, 1, 1</td>
<td>1</td>
<td>$\frac{6 \cdot 5}{2} = 15$</td>
<td>720</td>
</tr>
<tr>
<td>2, 1, 1, 1, 1</td>
<td>(1 2)</td>
<td>$\frac{6 \cdot 5 \cdot 4}{2} = 30$</td>
<td>720/15 = 48</td>
</tr>
<tr>
<td>2, 2, 1, 1</td>
<td>(1 2)(3 4)</td>
<td>$\frac{6 \cdot 5 \cdot 4 \cdot 3}{2} = 60$</td>
<td>720/45 = 16</td>
</tr>
<tr>
<td>2, 2, 2, 1</td>
<td>(1 2)(3 4)(5 6)</td>
<td>$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2} = 120$</td>
<td>720/15 = 48</td>
</tr>
<tr>
<td>3, 1, 1, 1, 1</td>
<td>(1 2 3)</td>
<td>$\frac{6 \cdot 5 \cdot 4 \cdot 3}{2} = 120$</td>
<td>720/40 = 18</td>
</tr>
<tr>
<td>3, 2, 1</td>
<td>(1 2 3)(4 5)</td>
<td>$\frac{6 \cdot 5 \cdot 4 \cdot 3}{2} \cdot \frac{2}{2} = 240$</td>
<td>720/120 = 6</td>
</tr>
<tr>
<td>3, 3</td>
<td>(1 2 3)(4 5 6)</td>
<td>$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2} = 240$</td>
<td>720/40 = 18</td>
</tr>
<tr>
<td>4, 1, 1</td>
<td>(1 2 3 4)</td>
<td>$\frac{6 \cdot 5 \cdot 4 \cdot 3}{2} = 120$</td>
<td>720/90 = 8</td>
</tr>
<tr>
<td>4, 2</td>
<td>(1 2 3 4)(5 6)</td>
<td>$\frac{6 \cdot 5 \cdot 4 \cdot 3}{2} \cdot \frac{2}{2} = 240$</td>
<td>720/90 = 8</td>
</tr>
<tr>
<td>5, 1</td>
<td>(1 2 3 4 5)</td>
<td>$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2} = 240$</td>
<td>720/144 = 5</td>
</tr>
<tr>
<td>6</td>
<td>(1 2 3 4 5 6)</td>
<td>$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2} = 240$</td>
<td>720/120 = 6</td>
</tr>
</tbody>
</table>

Exercise 2. [DF], §4.1, 7, Exercise 7. Here you will learn the notion of a primitive permutation representation. A group is called primitive if it has a faithful primitive permutation representation.

Solution.

(a) Clearly $1(B) = B$ so $1 \in G_B$. If $\sigma, \tau \in G_B$ then $\sigma \tau(B) = \sigma(\tau(B)) = \sigma(B) = B$, so $\sigma \tau \in G_B$. Finally if $\sigma \in G_B$, then $\sigma^{-1}(B) = \sigma^{-1}(\sigma(B)) = 1(B) = B$, so $\sigma^{-1} \in G_B$. This shows that $G_B$ contains 1 and is closed under multiplication and inverse. We conclude that $G_B$ is a group.

Suppose that $\sigma \in G_a$ and $a \in B$. Then $a = \sigma(a) \in \sigma(B)$ and $a \in \sigma(B) \cap B$. So $\sigma(B) \cap B \neq \emptyset$ and $\sigma(B) = B$ by the definition of a block. So $\sigma \in G_B$. We have shown that $G_a \subseteq G_B$.

(b) Suppose that $\sigma_i(B) \cap \sigma_j(B) \neq \emptyset$. Then

$$\sigma_i^{-1}(\sigma_i(B) \cap \sigma_j(B)) = B \cap (\sigma_i^{-1}\sigma_j)(B).$$
Either $B = \sigma_i^{-1}\sigma_j(B)$ and $\sigma_i(B) = \sigma_j(B)$ and $i = j$ or $B \cap \sigma_i^{-1}\sigma_j(B) = \emptyset$ and
\[\sigma_i(B) \cap \sigma_j(B) = \sigma_i(B \cap \sigma_i^{-1}\sigma_j(B)) = \sigma_i(\emptyset) = \emptyset.\]

This shows that $\sigma_1(B), \ldots, \sigma_n(B)$ are disjoint. Let $a \in B$. If $b \in A$ then there exists a $\sigma \in G$ such that $\sigma(a) = b$. It follows that $b \in \sigma(B)$. Since $\sigma_i(B) = \sigma_i(B)$ for some $i$ we see that $\sigma_1(B), \ldots, \sigma_n(B)$ is a partition of $A$.

(c) Consider $G = S_4$ acting on $\{1, 2, 3, 4\}$. A nontrivial block $B$ has size 2. (From part (b) follows that the size of a block is a divisor of $|A|$.) If $B$ is a block, then so is $\sigma(B)$. We can choose $\sigma$ such that $\sigma(B) = \{1, 2\}$. If we take $\tau = (2\ 3)$ then $\tau(\{1, 2\}) = \{2, 3\}$. But $\{1, 2\} \cap \tau(\{1, 2\}) = \{1, 2\} \cap \{2, 3\} = \{2\}$ is nonempty and $\{1, 2\} \neq \tau(\{1, 2\}) = \{2, 3\}$. This shows that $\{1, 2\}$ is not a block. So the action of $G = S_4$ is not primitive.

The group $D_8$ acts on the 4 vertices of a square. If $x, y$ are opposite vertices of the square, then so are $\sigma(x), \sigma(y)$ for any $\sigma \in D_8$. If we order the vertices of the square by 1, 2, 3, 4 by going around the square in counterclockwise fashion, then $\{1, 3\}$ and $\{2, 4\}$ are nontrivial blocks. So the action of $D_8$ on the vertices is not primitive.

(d) Suppose that the action is not primitive. Let $B$ be a nontrivial block (so $B$ has more than 1 element and $B \neq A$). Let $G_B$ be its stabilizer. The action of $G$ on $A$ is transitive, and the action of $G_B$ on $A$ is not transitive because the orbit of any $a \in B$ is contained in $B$ which is a strict subset of $A$. This shows that $G_B \neq G$. We have seen that $G_B$ contains $G_a$ for any $a \in G_B$. The group $G_B$ acts on $B$. We claim that this action is transitive: Suppose that $a, b \in B$. There exists an $\sigma \in G$ such that $\sigma(a) = b$. But then $b \in \sigma(B) \cap B$, so $B \cap \sigma(B) \neq \emptyset$ and we must have $B = \sigma(B)$ and $\sigma \in G_B$. The group $G_a$ does not act transitively on $B$ because the orbit of $a \in B$ is just $\{a\}$. This shows that $G_a \neq G_B$. So $G_B$ is a group strictly in between $G_a$ and $G$.

Suppose now that $H$ is a strict subgroup of $G$, strictly containing $G_a$. Define $B = Ha$ as the $H$-orbit of $a$. Clearly $B$ has more than 1 element because $H \neq G_a$. Suppose that $H$ acts transitively on $A$. Let $\sigma \in G$. Then there exists a $\tau \in H$ such that $\tau\sigma(a) = a$. So $\tau\sigma \in G_a \subseteq H$ and therefore $\sigma \in \tau^{-1}H = H$. This shows that $G = H$. Contradiction. Hence $H$ does not act transitively. Therefore $B = Ha \neq A$. We claim that $B$ is a block. Indeed, suppose that $\sigma(B) \cap B \neq \emptyset$ for some $\sigma \in G$. Say, $b \in \sigma(B) \cap B$. We can write $b = \tau_1(a)$ and $b = \sigma(\tau_2(a))$ for some $\tau_1, \tau_2 \in H$. So $\tau_1^{-1}\sigma\tau_2(a) = a$ and $\tau_1^{-1}\sigma\tau_2 \in G_a \subseteq H$. It follows that $\sigma \in H$ and $\sigma(B) = \sigma Ha = Ha = B$. So $B$ is a nontrivial block for the action of $G$. We conclude that the permutation representation is not primitive.

**Exercise 3.** [DF], §4.1, Exercise 9.
(a) We can write $O_i = H \cdot a_i$ for some $a_i \in A$. Suppose that $g \in G$. There exists a $j$ such that $ga_j \in O_j = Ha_j$. It follows that $gO_i = ghO_i = Hg(a_i) = Ha_j = O_j$. Since $G$ acts transitively on $A$, it acts transitively on $\{O_1, \ldots, O_r\}$. In particular for every $i \neq j$ there exists a $g$ such that $gO_i = O_j$. It follows that $|O_j| = |gO_i| = |O_i|$, so all orbits have the same cardinality.

(b) If $a \in O_1$ then $O_1 = H \cdot a$. The group $H$ acts transitively on $O_1$. So $O_1$ is the $H$-orbit of $a$. The stabilizer of $a$ (in $H$) is $H_a = G_a \cap H$. We have a natural bijection

$$H/H_a \cong H \cdot a = O_1.$$ 

From this follows that $|O_1| = |H : H \cap G_a|$. We have

$$r = \frac{|A|}{|O_1|} = \frac{|G|/|G_a|}{|H|/|H \cap G_a|} = \frac{|G|/|G_a|}{|G_a H|/|G_a|} = \frac{|G|}{|G_a H|} = |G : HG_a|.$$ 

**Exercise 4.** The group $S_n$ acts on $\{1, 2, \ldots, n\}$ as usual. Let $H$ be the stabilizer of $n$ (so $H$ is isomorphic to $S_{n-1}$). Describe the set of double cosets

$$H \backslash S_n / H = \{H \sigma H \mid \sigma \in S_n\}.$$ 

What is the cardinality of each double coset?

**Solution.** Double cosets in $H \backslash S_n / H$ are in one-to-one correspondence with $H$-orbits in $S_n / H$. There is a natural bijection $\psi : S_n / H \cong \{1, 2, \ldots, n\}$ that respects the action of $S_n$ ($S_n$ acts on $S_n / H$ by left multiplication). Here $H$ is the stabilizer of $n$ and $H \cong S_{\{1, 2, \ldots, n-1\}} = S_{n-1}$. So $\psi(H) = n$. Also $\psi((n-1 \ n)H) = n - 1$. There are clearly two $H$-orbits in $\{1, 2, \ldots, n\}$ namely $\{1, 2, \ldots, n-1\}$ and $\{n\}$. These two orbits correspond with the double cosets $H(n-1 \ n)H$ and $H1H = H$. Obviously $|H1H| = |H| = (n-1)!$. $H(n-1 \ n)H$ is the union of $n-1$ left cosets of $H$, so it has $(n-1)(n-1)!$ elements.

**Exercise 5.** Suppose that $G$ is a finite group and $H$ is a proper subgroup. Let $K_1, K_2, \ldots, K_r$ be the distinct conjugacy classes of $H$.

(a) Show that each $K_i$ is contained in a unique conjugacy class of $G$ (call it $L_i$).

**Solution.** Suppose that $K_i$ is the conjugacy class of $g_i$ in $H$. Let $L_i$ be the conjugacy class of $g_i$ in $G$. Clearly, if elements are conjugate in $H$ there are conjugate in $G$, so $K_i \subseteq L_i$.

(b) Prove that

$$|L_i| \leq |G : H||K_i|$$

where $|G : H| = |G|/|H|$ is the index of $H$ in $G$.

**Solution.** Suppose that

$$G/H = \{u_1H, u_2H, \ldots, u_rH\}$$
were \( r = |G : H| \). If \( l \in L_i \) then we can write \( l = kg_i k^{-1} \) for some \( k \in G \). Write \( k = u_j h \) for some \( h \in H \). then \( l = kg_i k^{-1} = u_j h g_i h^{-1} u_j^{-1} \in u_j K_i u_j^{-1} \). This shows that

\[
L_i \subseteq \bigcup_{i=1}^{r} u_i K_i u_i^{-1}.
\]

So we get

\[
|L_i| \leq r|L_i| = |G : H||L_i|.
\]

(c) Suppose that \( L_1, L_2, \ldots, L_r \) are all conjugacy classes of \( G \). Prove that \( |L_i| = |G : H||K_i| \) for all \( i \). Derive a contradiction by looking at the conjugacy class of 1.

Solution. Suppose that \( L_1, \ldots, L_r \) are all conjugacy classes (but some conjugacy classes may appear more than once). Choose indices \( i_1, i_2, \ldots, i_s \) such that \( L_{i_1}, \ldots, L_{i_s} \) are exactly all conjugacy classes, without repetition. Then

\[
|G| = |L_{i_1}| + |L_{i_2}| + \cdots + |L_{i_s}| \leq |G : H|\left(|K_{i_1}| + |K_{i_2}| + \cdots + |K_{i_s}|\right) \leq |G : H|\left(|K_1| + \cdots + |K_r|\right) = |G : H||H| = |G|.
\]

So all the inequalities above are equalities. It follows that \( \{i_1, \ldots, i_s\} = \{1, 2, \ldots, r\} \) and \( r = s \). Moreover, we must have \( |L_{i_j}| = |G : H||K_{i_j}| \) for all \( j \), so \( |L_i| = |G : H||K_i| \) for all \( i \). For some \( i \), \( L_i \) is the conjugacy class of 1, and \( K_i \) must be the conjugacy class of 1 in \( H \). But then \( 1 = |L_i| = |G : H||K_i| = |G : H| \), so \( |G : H| = 1 \) and \( H = G \). Contradiction!

We have now proven the following theorem:

**Theorem:** If \( G \) is a finite group and \( H \) is a proper subgroup, then there exists a conjugacy class \( L \) of \( G \) such that \( L \cap H = \emptyset \).

**Hard Exercises (optional, for extra credit)**

**Exercise 6.** [DF], §4.2, Exercise 8. (Not really hard, but you need the right idea.)

Solution. Consider the action of \( G \) on \( G/H \). Let \( K \) be the kernel of this action. Then \( G/K \) acts faithfully on \( G/H \). So \( G/K \) is a subgroup of \( S_{G/H} \cong S_n \). So \( |G : K| = |G/K| \leq |S_n| = n! \). (Note: \( K = \bigcap_{g \in G} gHg^{-1} \).)

**Exercise 7.** [DF], §4.3, Exercise 26.

Solution. Define \( S = \{(g, a) \in G \times A \mid g \cdot a = a\} \). For any \( a \in A \), \( A \cong G/G_a \), so \( |G_a| = |G|/|A| \). Since

\[
S = \bigcup_{a \in A} G_a \times \{a\}
\]
we have
\[ |S| = \sum_{a \in A} |G_a| = |A| \frac{|G|}{|A|} = |G|. \]

For \( g \in G \), define \( \text{Fix}(g) = \{ a \in A \mid g \cdot a = a \} \). We have
\[ S = \bigcup_{g \in G} \{ g \} \times \text{Fix}(g) \]
so
\[ |G| = |S| = \sum_{g \in G} |\text{Fix}(g)|. \]

So “on average”, \( \text{Fix}(g) \) has 1-element. Note that \( \text{Fix}(1) = |A| > 1 \). So
\[ \sum_{g \in G \setminus \{1\}} |\text{Fix}(g)| < |G| - 1. \]

By the pigeonhole principle, there must exist a \( g \in G \setminus \{1\} \) such that \( |\text{Fix}(g)| = 0 \), i.e., \( \text{Fix}(g) = \emptyset \).

**Exercise 8.** [DF], §4.3, Exercise 27. (The only proof that I can see uses Exercise 5.)

**Solution.** Let \( H \) be the group generated by \( g_1, \ldots, g_r \). This group is abelian. Every conjugacy class of \( G \) intersects with \( H \). From Exercise 5 follows that \( H = G \). Thus, \( G \) is abelian.