MATH 594, WINTER 2006, PROBLEM SET 3

DUE: **WEDNESDAY**, 2/8/2006

For more on the classification of simple groups, see the Notices article:
`http://www.ams.org/notices/199502/solomon.pdf`

Warm-up (not to be handed in)

[**DF**], §4.5, exercises 30, 31, 33, 40, §4.6, exercises 1,2,3, §3.4, exercises 5, 7, §1.2, exercise 12.

Exercises to be handed in

Exercise 1. Do [**DF**], §4.5, Exercise 34.

Solution. Let Q be a p-sylow subgroup of N. Then there exists a $g \in G$ such that $Q \subseteq gPg^{-1}$. So $g^{-1}Qg \subseteq P$. Since N is normal, $g^{-1}Qg \subseteq g^{-1}Ng = N$, and $g^{-1}Qg \subseteq P \cap N$. Now $g^{-1}Qg$ is a p-Sylow subgroup of N because $|g^{-1}Qg| = |Q|$. The group $P \cap N$ is a p-subgroup of N. We must have $g^{-1}Qg = P \cap N$. This shows that $P \cap N$ is a p-Sylow subgroup of N. Suppose that $|P| = p^{\alpha}$ and $|Q| = p^{\beta}$. Then a p-Sylow subgroup of G/N has order $p^{\alpha-\beta}$. We have

$$PN/N \cong P/(P \cap N)$$

So $|PN/N| = |P|/|P \cap N| = p^{\alpha}/p^{\beta} = p^{\alpha-\beta}$ and PN/N is a p-Sylow subgroup of G/N.

Exercise 2. Do [**DF**], §4.5, Exercise 42. (*Hint:* The icosahedron has 12 vertices, 30 edges and 20 faces. Find a set of 6 elements on which this group of rigid motions (rotations) acts faithfully. You may assume that the group has 60 elements.)

Solution. This problem is a bit hard. The Icosahedron group, let us call it I, has 60 elements. (To see this: the stabilizer of a vertex has order 5, and the action on the set of 12 vertices is transitive. The group order therefore is $5 \cdot 12 = 60$.) Consider the 6 diagonals that connect two antipodal vertices. The group acts on the 6 diagonals. This action is faithful: Suppose that $g \in I$ stabilizes all diagonals. This means that g sends every vertex to itself or to its antipode. It is easy to see that if such a g fixes one vertex, it must also fix all its neigbors. From this one sees that an element of g that fixes all diagonals must be $\pm$Id. Now g cannot be $-$Id because the group of rigid motions only contains rotations (i.e., isometries with determinant $+1$).

We rather would have a faitful actions on a set of 5 elements. To get this, we will define a subgroup of I of index 5. Let us label the diagonals. Or rather, we label the vertices, but we give antipodal vertices the same label. Take a triangular face and label the vertices $1, 2, 3$ (say counterclockwise). Now 1 is connected to a unique vertex which is not 2 or 3 and which is not connected to 2 or 3. Label this vertex 4. Similarly, label the vertex connected to 2, "opposite" the edge 13 by 5 and the vertex connected to 3 "opposite" the edge 12 by 6. All the other vertices are antipodal to these 6 vertices. If we rotate the face 123, then we also rotate $4, 5, 6$. This shows that $(1\ 2\ 3)(4\ 5\ 6) \in I$. If we send 1 and 4 to their antipodals, then this element interchanges 2 and 5 and interchanges 3 and 6. So $(2\,5)(3\,6) \in I$. Let H be the subgroup generated by $(1\ 2\ 3)(4\ 5\ 6)$ and $(2\ 5)(3\ 6)$. One can easily see that H contains the subgroup $V = \{1, (2\ 5)(3\ 6), (1\ 4)(2\ 5), (1\ 4)(3\ 6)\}$ of order 4. It is also easy to see that $(1\ 2\ 3)(4\ 5\ 6)$ normalizes V. This shows that $H \cong Z_4 \rtimes Z_3$. In particular H has order 12 and index 5. So I has a transitive action on I/H and $|I/H| = 5$. We need to check that this action is faithful. It suffices to show that $\bigcap_{g \in I} gHg^{-1} = \{1\}$. For example, the group $H' = \langle (2\ 3\ 4)(6\ 5\ 1), (2\ 6)(3\ 5) \rangle$ is conjugate and $H' \cap H = \{1\}$. This allows us to view I as a subgroup of S_5 of index 2. So I is a normal subgroup of S_5. Since A_5 is simple, the homomorphism $A_5 \to S_5/I$ must be trivial, and $A^5 \subseteq I$. We conclude that $I \cong A_5$.

Exercise 3. Do **[DF]**, §4.5, Exercise 46.

Solution. Note that $|S_{p^2}| = (p^2)!$. Among the numbers $1, 2, 3, \ldots, p^2$, only $p, 2p, 3p, \ldots, p^2$ are divisible by p. Among $p, 2p, \ldots, p^2$, only p^2 is divisible by p^2. So the number of factors p is $p + 1$ and every p-Sylow subgroup of S_{p^2} has order p^{p+1}. It is easy to identify a abelian subgroup H of S_{p^2} of order p^p. Namely, let H be the subgroup generated by the p p-cycles

$$(1\ 2\ \cdots p), (p+1\ p+2\ \ldots\ 2p), \ldots, (p^2-p+1\ p^2-p+2\ \cdots\ p^2).$$

Note that the element σ defined by

$$\sigma = (1\ p+1\ 2p+1\ \cdots p^2-p+1)(2\ p+2\ 2p+2\ \cdots\ p^2-p+2)\cdots(p\ 2p\ 3p\ \cdots p^2)$$

normalizes H. Het K be the subgroup generated by σ. Then K has order p. Clearly $H \cap K = \{1\}$. This shows that HK is a group of order $|H||K|/|H \cap K| = p^{p+1}$. So HK is a p-Sylow subgroup. (It is a wreath product!)

Exercise 4.

(a) Do **[DF]**, §4.5, Exercise 1.
Solution. A p-subgroup P of a group G is a p-Sylow subgroup if p does not divide the index $|G : P|$. Suppose that P is a p-Sylow subgroup of G and P is a subgroup of H. Since $|H : P|$ divides $|G : P|$, p does not divide $|H : P|$. So P is a p-Sylow subgroup of H.

(b) Do **[DF]**, §4.5, Exercise 2.
Solution. $|gHg^{-1} : gQg^{-1}| = |gHg^{-1}|/|gQg^{-1}| = |H|/|Q| = |H : Q|$, so p does not divide $|gHg^{-1} : gQg^{-1}|$, and gQg^{-1} is a p-subgroup of gHg^{-1}. Therefore, gQg^{-1} is a p-Sylow subgroup of gHg^{-1}.

(c) Do **[DF]**, §4.5, Exercise 3. *Solution.* Suppose that $|G|$ is divisible by p. Let P be a p-Sylow subgroup. Then P is nontrivial. Let $g \in P$ such that $g \neq 1$. Then the order of g divides $|P|$. Let p^α be the order of g. Define $h = g^{p^{\alpha-1}}$. Then h has order p.

Exercise 5. Suppose that q is a positive power of a prime p. Consider the subgroup U of upper triangular matrices with 1's on the diagonal inside $G := \mathrm{GL}_n(\mathbb{F}_q)$ (I have called this group B before by mistake, typically this subgroup is denoted by U, for *unipotent* or by N for *nilpotent*). We have seen that U is a p-Sylow subgroup. How many p-Sylow subgroups does $\mathrm{GL}_n(\mathbb{F}_q)$ have? (*Hint:* Compute $N_G(U)$.)

Solution. Let V_r be the set of all vectors

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

with $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-r} = 0$. So we have

$$\mathbb{F}_q^n = V_n \supset V_{n-1} \supset \cdots \supset V_0 = \{0\}.$$

It is easy to see that V_1 is the fixed point set of the action of U on $\mathbb{F}_q^n$.

Similarly one can see that

$$V_{i+1} = \{v \in \mathbb{F}_q^n \mid (A - I)v \in V_i \text{ for all } A \in U\}.$$

Suppose that $C \in N_G(U)$. By induction on i we show that $C(V_i) \subseteq V_i$. The case $i = 0$ is clear. Suppose that $A \in U$ and $v \in V_{i+1}$. Then $(C^{-1}AC - I)v \in V_i$ because $C^{-1}AC \in U$. So $(A - I)(Cv) = (AC - C)v = C((C^{-1}AC - I)v) \in V_i$ because $C(V_i) \subseteq V_i$ by induction. Since this is true for arbitrary $A \in U$, we get that $Cv \in V_{i+1}$. This shows that $C(V_{i+1}) \subseteq V_{i+1}$.

So we see that $C(V_i) \subseteq V_i$ for all i. This implies that C is an upper triangular matrix. We conclude that $N_G(U)$ is contained in B, where B is the set of invertible upper triangular matrices. On the other hand, it is easy to see that B normalizes U. We can identify $\mathrm{Syl}_p(G)$ with $G/N_G(U) = G/B$. We have

$$|G| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$$

and

$$|B| = (q - 1)^n q^{\binom{n}{2}}.$$

From this follows that the number of p-Sylow subgroups is

$$\frac{|G|}{|B|} = \frac{(q^n-1)(q^{n-1}-1)\cdots(q-1)}{(q-1)^n} =$$

$$= (q^{n-1}+q^{n-2}+\cdots+q+1)(q^{n-2}+q^{n-3}+\cdots+q+1)\cdots(q^2+q+1)(q+1).$$

Hard Exercises (optional, for extra credit)

Exercise 6. Prove that $\mathrm{PSL}_3(\mathbb{F}_2) = \mathrm{GL}_3(\mathbb{F}_2)$ is a simple group with $168 = 2^3 \cdot 3 \cdot 7$ elements.

Solution. Suppose that $A \in G := \mathrm{GL}_3(\mathbb{F}_2)$ has order 2. Then A is unipotent; $(A-I)^2 = A^2 - I = 0$. From linear algebra we know that such an element must be conjugate to

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Suppose that N is a nontrivial normal subgroup of G. By Cauchy's theorem, N contains an element of order 2, 3 or 7. Suppose that N has an element of order 2. Then N contains

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

These two matrices generate the upper triangular matrices (which is a 2-Sylow subgroup of order 8). We know that the number of 2-Sylow subgroups in G is $(2^2+2+1)(2+1) = 21$. So N has also 21 distinct 2-Sylow subgroups. But then $|N|$ is divisible by 21 and by 8 so we must have $N = G$.

Suppose N contains an element of order 7. All 7-Sylow subgroups are conjugate. So N contains all elements of order 7. In particular N contains

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

which both have order 7 and their product

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

which as order 2. But then $N = G$ as we have seen.

Suppose that N has an element of order 3. Then N contains all elements of order 3. So N contains

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and their product

$$CD = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

has order 7. But then $N = G$ as we have seen.

We conclude that $N = G$ and G is simple.

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http://www.math.lsa.umich.edu/~hderksen/math594.w06/index.html