For more on the classification of simple groups, see the Notices article:

Warm-up (not to be handed in)

[DF], §4.5, exercises 30, 31, 33, 40, §4.6, exercises 1, 2, 3, §3.4, exercises 5, 7, §1.2, exercise 12.

Exercises to be handed in

Exercise 1. Do [DF], §4.5, Exercise 34.

Solution. Let $Q$ be a $p$-sylow subgroup of $N$. Then there exists a $g \in G$ such that $Q \subseteq gPg^{-1}$. So $g^{-1}Qg \subseteq P$. Since $N$ is normal, $g^{-1}Qg \subseteq g^{-1}Ng = N$, and $g^{-1}Qg \subseteq P \cap N$. Now $g^{-1}Qg$ is a $p$-Sylow subgroup of $N$ because $|g^{-1}Qg| = |Q|$. The group $P \cap N$ is a $p$-subgroup of $N$. We must have $g^{-1}Qg = P \cap N$. This shows that $P \cap N$ is a $p$-Sylow subgroup of $N$. Suppose that $|P| = p^\alpha$ and $|Q| = p^\beta$. Then a $p$-Sylow subgroup of $G/N$ has order $p^{\alpha-\beta}$. We have

$$PN/N \cong P/(P \cap N)$$

So $|PN/N| = |P|/|P \cap N| = p^\alpha/p^\beta = p^{\alpha-\beta}$ and $PN/N$ is a $p$-Sylow subgroup of $G/N$.

Exercise 2. Do [DF], §4.5, Exercise 42. (Hint: The icosahedron has 12 vertices, 30 edges and 20 faces. Find a set of 6 elements on which this group of rigid motions (rotations) acts faithfully. You may assume that the group has 60 elements.)

Solution. This problem is a bit hard. The Icosahedron group, let us call it $I$, has 60 elements. (To see this: the stabilizer of a vertex has order 5, and the action on the set of 12 vertices is transitive. The group order therefore is $5 \cdot 12 = 60$.) Consider the 6 diagonals that connect two antipodal vertices. The group acts on the 6 diagonals. This action is faithful: Suppose that $g \in I$ stabilizes all diagonals. This means that $g$ sends every vertex to itself or to its antipode. It is easy to see that if such a $g$ fixes one vertex, it must also fix all its neighbors. From this one sees that an element of $g$ that fixes all diagonals must be $\pm \text{Id}$. Now $g$ cannot be $-\text{Id}$ because the group of rigid motions only contains rotations (i.e., isometries with determinant +1).
We rather would have a faithful actions on a set of 5 elements. To get this, we will define a subgroup of $I$ of index 5. Let us label the diagonals. Or rather, we label the vertices, but we give antipodal vertices the same label. Take a triangular face and label the vertices 1, 2, 3 (say counterclockwise). Now 1 is connected to a unique vertex which is not 2 or 3 and which is not connected to 2 or 3. Label this vertex 4. Similarly, label the vertex connected to 2, “opposite” the edge 13 by 5 and the vertex connected to 3 “opposite” the edge 12 by 6. All the other vertices are antipodal to these 6 vertices. If we rotate the face 123, then we also rotate 4, 5, 6. This shows that $(123)(456) \in I$. If we send 1 and 4 to their antipodals, then this element interchanges 2 and 5 and interchanges 3 and 6. So $(25)(36) \in I$.

Let $H$ be the subgroup generated by $(123)(456)$ and $(25)(36)$. One can easily see that $(123)(456)$ normalizes $V$. This shows that $H \cong \mathbb{Z}_4 \times \mathbb{Z}_3$. In particular $H$ has order 12 and index 5. So $I$ has a transitive action on $I/H$ and $|I/H| = 5$. We need to check that this action is faithful. It suffices to show that $\bigcap_{g \in I} gHg^{-1} = \{1\}$. For example, the group $H' = \{(234)(651), (26)(35)\}$ is conjugate and $H' \cap H = \{1\}$. This allows us to view $I$ as a subgroup of $S_5$ of index 2. So $I$ is a normal subgroup of $S_5$. Since $A_5$ is simple, the homomorphism $A_5 \rightarrow S_5/I$ must be trivial, and $A_5 \subseteq I$. We conclude that $I \cong A_5$.

**Exercise 3.** Do [DF], §4.5, Exercise 46.

**Solution.** Note that $|S_{p^2}| = (p^2)!$. Among the numbers 1, 2, 3, ..., $p^2$, only $p, 2p, 3p, \ldots, p^2$ are divisible by $p$. Among $p, 2p, \ldots, p^2$, only $p^2$ is divisible by $p^2$. So the number of factors $p$ is $p + 1$ and every $p$-Sylow subgroup of $S_{p^2}$ has order $p^{p+1}$. It is easy to identify a abelian subgroup $H$ of $S_{p^2}$ of order $p^9$. Namely, let $H$ be the subgroup generated by the $p$-cycles

$$(1 \ 2 \ \cdots \ p), (p + 1 \ p + 2 \ \cdots \ 2p), \ldots, (p^2 - p + 1 \ p^2 - p + 2 \ \cdots \ p^2).$$

Note that the element $\sigma$ defined by

$$\sigma = (1 \ p + 1 \ 2p + 1 \ \cdots \ p^2 - p + 1)(2 \ p + 2 \ 2p + 2 \ \cdots \ p^2 - p + 2) \cdots (p \ 2p \ 3p \ \cdots \ p^2)$$

normalizes $H$. Let $K$ be the subgroup generated by $\sigma$. Then $K$ has order $p$. Clearly $H \cap K = \{1\}$. This shows that $HK$ is a group of order $|H||K|/|H \cap K| = p^{p+1}$. So $HK$ is a $p$-Sylow subgroup. (It is a wreath product!)

**Exercise 4.**

(a) Do [DF], §4.5, Exercise 1.

**Solution.** A $p$-subgroup $P$ of a group $G$ is a $p$-Sylow subgroup if $p$ does not divide the index $|G : P|$. Suppose that $P$ is a $p$-Sylow subgroup of $G$ and $P$ is a subgroup of $H$. Since $|H : P|$ divides $|G : P|$, $p$ does not divide $|H : P|$. So $P$ is a $p$-Sylow subgroup of $H$. 
GL is denoted by \( U \).

Suppose that \( \alpha \in G \).

Exercise 5.

(a) Suppose that \( \alpha \in G \) is divisible by \( p \).

Solution. \(|gHg^{-1} : gQg^{-1}| = |gHg^{-1}| / |gQg^{-1}| = |H| / |Q| = |H : Q|\), so \( p \) does not divide \(|gHg^{-1} : gQg^{-1}|\), and \( gQg^{-1} \) is a \( p \)-subgroup of \( gHg^{-1} \). Therefore, \( gQg^{-1} \) is a \( p \)-Sylow subgroup of \( gHg^{-1} \).

(b) Do [DF], §4.5, Exercise 2.

Solution.

(c) Do [DF], §4.5, Exercise 3. Solution. Suppose that \( |G| \) is divisible by \( p \).

Let \( P \) be a \( p \)-Sylow subgroup. Then \( P \) is nontrivial. Let \( g \in P \) such that \( g \neq 1 \). Then the order of \( g \) divides \( |P| \). Let \( p^a \) be the order of \( g \). Define \( h = g^{p^{a-1}} \). Then \( h \) has order \( p \).

Exercise 5. Suppose that \( q \) is a positive power of a prime \( p \). Consider the subgroup \( U \) of upper triangular matrices with 1’s on the diagonal inside \( G := \text{GL}_n(\mathbb{F}_q) \) (I have called this group \( B \) before by mistake, typically this subgroup is denoted by \( U \), for unipotent or by \( N \) for nilpotent). We have seen that \( U \) is a \( p \)-Sylow subgroup. How many \( p \)-Sylow subgroups does \( \text{GL}_n(\mathbb{F}_q) \) have? (Hint: Compute \( N_G(U) \).)

Solution. Let \( V_r \) be the set of all vectors

\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{pmatrix}
\]

with \( \alpha_1, \ldots, \alpha_n \in \mathbb{F}_q \) and \( \alpha_1 = \alpha_2 = \cdots = \alpha_{n-r} = 0 \). So we have

\[
\mathbb{F}_q^n = V_n \supset V_{n-1} \supset \cdots \supset V_0 = \{0\}.
\]

It is easy to see that \( V_1 \) is the fixed point set of the action of \( U \) on \( \mathbb{F}_q^n \).

Similarly one can see that

\[
V_{i+1} = \{ v \in \mathbb{F}_q^n \mid (A - I)v \in V_i \text{ for all } A \in U \}.
\]

Suppose that \( C \in N_G(U) \). By induction on \( i \) we show that \( C(V_i) \subseteq V_i \). The case \( i = 0 \) is clear. Suppose that \( A \in U \) and \( v \in V_{i+1} \). Then \((C^{-1}AC - I)v \in V_i \) because \( C^{-1}AC \in U \). So \((A - I)(Cv) = (AC - C)v = C((C^{-1}AC - I)v) \in V_i \) because \( C(V_i) \subseteq V_i \) by induction. Since this is true for arbitrary \( A \in U \), we get that \( Cv \in V_{i+1} \). This shows that \( C(V_{i+1}) \subseteq V_{i+1} \).

So we see that \( C(V_i) \subseteq V_i \) for all \( i \). This implies that \( C \) is an upper triangular matrix. We conclude that \( N_G(U) \) is contained in \( B \), where \( B \) is the set of invertible upper triangular matrices. On the other hand, it is easy to see that \( B \) normalizes \( U \). We can identify \( \text{Syl}_p(G) \) with \( G/N_G(U) = G/B \). We have

\[
|G| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})
\]

and

\[
|B| = (q - 1)^n q^{\binom{n}{2}}.
\]
From this follows that the number of $p$-Sylow subgroups is
\[
\frac{|G|}{|B|} = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}{(q - 1)^n} =
\]
\[
= (q^{n-1} + q^{n-2} + \cdots + q + 1)(q^{n-2} + q^{n-3} + \cdots + q + 1) \cdots (q^2 + q + 1)(q + 1).
\]

**Hard Exercises (optional, for extra credit)**

**Exercise 6.** Prove that $\text{PSL}_3(\mathbb{F}_2) = \text{GL}_3(\mathbb{F}_2)$ is a simple group with $168 = 2^3 \cdot 3 \cdot 7$ elements.

*Solution.* Suppose that $A \in G := \text{GL}_3(\mathbb{F}_2)$ has order 2. Then $A$ is unipotent; $(A - I)^2 = A^2 - I = 0$. From linear algebra we know that such an element must be conjugate to
\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Suppose that $N$ is a nontrivial normal subgroup of $G$. By Cauchy’s theorem, $N$ contains an element of order 2, 3 or 7. Suppose that $N$ has an element of order 2. Then $N$ contains
\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]
These two matrices generate the upper triangular matrices (which is a 2-Sylow subgroup of order 8). We know that the number of 2-Sylow subgroups in $G$ is $(2^2 + 2 + 1)(2 + 1) = 21$. So $N$ has also 21 distinct 2-Sylow subgroups. But then $|N|$ is divisible by 21 and by 8 so we must have $N = G$.

Suppose $N$ contains an element of order 7. All 7-Sylow subgroups are conjugate. So $N$ contains all elements of order 7. In particular $N$ contains
\[
A = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
B = \begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]
which both have order 7 and their product
\[
AB = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]
which as order 2. But then $N = G$ as we have seen.
Suppose that $N$ has an element of order 3. Then $N$ contains all elements of order 3. So $N$ contains

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and their product

$$CD = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

has order 7. But then $N = G$ as we have seen.

We conclude that $N = G$ and $G$ is simple.

Harm Derksen, 3067EH, 763 2309

Office hours: **MWF 3-4pm (changed)**.

http://www.math.lsa.umich.edu/~hderksen/math594.w06/index.html