MATH 594, WINTER 2006, FINAL EXAM

DUE: **MONDAY**, 4/20/2006

(1) (2,2,2,2 pts.) Let $\alpha = \sqrt{2+\sqrt{5}}$.
   (a) What is the minimum polynomial $f(x)$ of α over $\mathbb{Q}$?
   (b) Show that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is *not* a Galois extension.
   (c) Let E be a splitting field of $f(x)$. Determine the Galois group $\mathrm{Gal}(E/\mathbb{Q})$.
   (d) Prove that there is a unique subfield K of E such that $[K : \mathbb{Q}] = 4$ and $K/\mathbb{Q}$ is Galois. What is K?

(2) (2,2,2,2 pts.) Let ζ be a primitive 7-th root of unity.
   (a) Describe a generator of the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.
   (b) Let $\alpha = \zeta + \zeta^2 - \zeta^3 + \zeta^4 - \zeta^5 - \zeta^6$. Show that $\alpha^2 = -7$. (Hint: set $\beta = \zeta + \zeta^2 + \zeta^4$ and verify that $\beta^2 = -\beta - 2$.)
   (c) What is $\mathrm{Gal}(\mathbb{Q}(\zeta, \sqrt{2})/\mathbb{Q})$?
   (d) Show that $\mathbb{Q}(\zeta, \sqrt{2}) = \mathbb{Q}(\zeta + \sqrt{2})$.

(3) (4 pts.) Suppose that F is a field such that every irreducible polynomial with coefficients in F has odd degree. Prove that every polynomial with coefficients in F can be solved by radicals. (You may use here that simple groups of odd order are abelian.)