

**HOMEWORK 2 ON CHAPTER I, MOSTLY §2**  
**ALGEBRAIC GEOMETRY I**  
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- (1) Suppose that  $f(x, y)$  and  $g(x, y)$  are irreducible polynomials. Let

$$X = \{(x, y) \in k^2 \mid f(x, y)\} \quad \text{and}$$

$$Y = \{(x, y) \in k^2 \mid g(x, y) = 0\}.$$

Suppose that  $X$  and  $Y$  intersect at  $P \in \mathbb{A}^2$ . Assume that  $X$  and  $Y$  are both nonsingular at  $P$ . Prove that the following statements are equivalent:

- (a) The intersection multiplicity of  $X$  and  $Y$  at  $P$  is 1.
- (b)  $g(x, y)$  is a local parameter for  $X$  at  $P$ .
- (c)  $f(x, y)$  is a local parameter for  $Y$  at  $P$ .
- (d)

$$\begin{vmatrix} f'_x(P) & f'_y(P) \\ g'_x(P) & g'_y(P) \end{vmatrix} \neq 0.$$

- (2) Let  $f(x) = p(x)/q(x)$  be a rational function on  $\mathbb{A}^1$ , with  $p(x), q(x) \in k[x]$ . It extends to a rational function on  $\mathbb{P}^1$  via the inclusion

$$x \in \mathbb{A}^1 \mapsto (x : 1) \in \mathbb{P}^1.$$

- (a) What is the multiplicity of  $f(x)$  at  $\infty = [1 : 0]$ ? (in terms of  $\deg p, \deg q$ )
- (b) Prove the formula

$$\sum_{P \in \mathbb{P}^1} m_P = 0,$$

where  $m_P$  is the multiplicity of  $f$  at  $P \in \mathbb{P}^1$ .

- (c) Show that if  $f$  is regular at every point of  $\mathbb{P}^1$ , then  $f$  is constant.
- (3) Consider  $\mathbb{A}^n$  as a topological space with the Zariski topology. We already saw that  $\mathbb{A}^n$  is not Hausdorff.
- (a) Prove that  $\mathbb{A}^n$  is Noetherian as a topological space. This means that if  $X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$ , then we must have  $X_m = X_{m+1} = \dots$  for some positive integer  $m$ .
  - (b) Is  $\mathbb{A}^n$  quasi-compact? In other words: Suppose that  $U_\gamma, \gamma \in S$  is an open covering of  $\mathbb{A}^n$  indexed by the (possibly infinite) set  $S$ , so  $\bigcup_{\gamma \in S} U_\gamma = \mathbb{A}^n$ . Can we always choose a finite subset  $T \subseteq S$  such that  $U_\gamma, \gamma \in T$  is a covering of  $\mathbb{A}^n$  as well?
  - (c) Is the Zariski topology of  $\mathbb{A}^{n+m}$  equal to the product topology of  $\mathbb{A}^n$  and  $\mathbb{A}^m$ ?

- (d) Show that a regular map  $\phi : \mathbb{A}^n \rightarrow \mathbb{A}^m$  is continuous.
- (e) Is the projection  $p : \mathbb{A}^2 \rightarrow \mathbb{A}^1$  onto the first coordinate a closed map? Motivate your answer. (Recall that a map between topological spaces is called closed (resp. open) if it maps closed sets to closed sets (resp. open sets to open sets).)
- (f) Is the projection  $q : \mathbb{A}^n \times \mathbb{A}^m \rightarrow \mathbb{A}^n$  an open map? Prove your answer.
- (4) (a) Suppose that  $X \subset \mathbb{A}^n$  is a closed set and suppose that  $f \in K[X]$ . Let  $U = X \setminus \{f = 0\}$  be the open set where  $f$  does not vanish. We can identify  $U$  with the set

$$V := \{(x, 1/f(x)) \in \mathbb{A}^n \times \mathbb{A}^1 \mid x \in U\}.$$

- (b) Show that  $V \subset \mathbb{A}^{n+1}$  is closed. Show that the coordinate ring  $k[V]$  is isomorphic to  $R[T]/(Tf - 1)$ , where  $R = K[X]$  and  $T$  is an indeterminate. (We define  $k[U] := k[V]$ .)
- (c) Let  $\mathrm{GL}_n(k) \subset M_n(k) \cong \mathbb{A}^{n^2}$  be the group of invertible matrices inside the set of  $n \times n$  matrices with entries in  $k$ . Then  $\mathrm{GL}_n(k)$  is an open set defined by the non-vanishing of the determinant. As before, we may view  $\mathrm{GL}_n(k)$  as a closed subset in  $\mathbb{A}^{n^2+1}$ . Prove that the multiplication map  $m : \mathrm{GL}_n(k) \times \mathrm{GL}_n(k) \rightarrow \mathrm{GL}_n(k)$  and the inverse map  $i : \mathrm{GL}_n(k) \rightarrow \mathrm{GL}_n(k)$  are regular maps.
- (d) A Zariski closed subgroup of  $\mathrm{GL}_n(k)$  (for some  $n$ ) is called a linear algebraic group. Prove that the orthogonal group is a linear algebraic group. Also show that any finite group can be seen as a linear algebraic group.
- (5) (Bonus) It is not so easy to see if two affine varieties are isomorphic. In this problem, 1 star means hard, 2 stars means unreasonably hard, and 3 stars means that it is an important open problem in algebraic geometry. We work over the field  $k = \mathbb{C}$ .
- (a) \* Define  $X, Y \subseteq \mathbb{A}^3$  by

$$X = \{(x, y, z) \in \mathbb{A}^3 \mid xy - z^2 = 1\},$$

$$Y = \{(x, y, z) \in \mathbb{A}^3 \mid x^2y - z^2 = 1\}.$$

Prove that  $X \times \mathbb{A}^1$  and  $Y \times \mathbb{A}^1$  are isomorphic.

- (b) \*\* Prove that  $X$  and  $Y$  are *not* isomorphic.
- (c) \*\*\* Prove or disprove: If  $X \times \mathbb{A}^1$  is isomorphic to  $\mathbb{A}^{n+1}$  then  $X$  is isomorphic to  $\mathbb{A}^n$ . (This is the so-called Zariski cancellation problem.)
- (6) About generalizations of the Abhyankar-Moh-Suzuki theorem. Suppose that  $X \subseteq \mathbb{A}^n$  is isomorphic to  $\mathbb{A}^1$ .
- (a) Suppose that  $X \subseteq \mathbb{A}^{n-1} \times \{0\} \subseteq \mathbb{A}^n$ . Prove that there exists an automorphism  $\varphi$  of  $\mathbb{A}^n$  such that  $\varphi(X)$  is a line.
- (b) (without the assumption of the previous part) For  $n \geq 4$  it is known that after a linear change of coordinates, one may assume that the projection  $p : \mathbb{A}^n \rightarrow \mathbb{A}^{n-1}$  induces an isomorphism of

- $X$  with its image  $p(X)$  in  $\mathbb{A}^{n-1}$ . Use this fact to prove that there exists an automorphism  $\varphi$  of  $\mathbb{A}^n$  such that  $\varphi(X)$  is a line.
- (c) \*\*\* Assume  $n = 3$ . Does there always exist an automorphism  $\varphi$  of  $\mathbb{A}^3$  such that  $\varphi(X)$  is a line.