HOMEWORK 3 ON CHAPTER I
ALGEBRAIC GEOMETRY I
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Note: As usual, we will work over an algebraically closed base field \( k \).

**Extended Deadline!** As a matter of policy I will give you two weeks
deadline for each problem set from now on. This will give you more chance
to ask questions. However, I still plan to produce problem sets on a weekly
basis.

(1) (a) Suppose that \( C \) is a plane curve such that \( C \times \mathbb{A}^{n-1} \) is birational
to \( \mathbb{A}^n \). Prove that \( C \) is birational to \( \mathbb{A}^1 \).
(b) * Suppose that \( p : \mathbb{A}^2 \rightarrow \mathbb{A} \) is a regular, non-constant map.
Prove that \( p^{-1}(\alpha) \) is smooth for some \( \alpha \in \mathbb{A} \). (Upgraded to a
**bonus problem.**) You have to assume that \( k \) has characteristic
0. Can you find a counterexample if \( k \) has positive characteris-
tic? (Hint: The tricky case is when \( p'_x \) and \( p'_y \) have a common
irreducible factor, say \( q \). View \( p \) as a function on \( q(x, y) = 0 \)
and prove that \( p \) must be constant.)

(c) Suppose that \( C \) is a plane curve with \( C \times \mathbb{A}^{n-1} \cong \mathbb{A}^2 \). Assume that
the characteristic is 0. (Once we have introduced tangent spaces
of closed sets, it will be easy to prove that \( C \) is smooth, also
in positive characteristic.) Prove that \( C \) is a rational smooth
curve and that \( k[C]^* = k^* \). (Recall that

\[
R^* = \{ f \in R \mid \exists g \in R \text{ } fg = 1 \}
\]

is the set of invertible elements in \( R \) for any ring \( R \).

(d) Let \( C \) be a curve as in (c). Prove that \( C \) is isomorphic to \( \mathbb{A}^1 \).
(You could show first that there exists a regular birational map
\( \varphi : C \rightarrow \mathbb{A}^1 \).

Hence we have proven a special case of the Zariski Cancellation
Problem: If \( C \) is an affine curve and \( C \times \mathbb{A}^1 \cong \mathbb{A}^2 \), then \( C \cong \mathbb{A}^1 \).

(2) An affine algebraic group is a Zariski closed set \( G \) together with:
(1) an element \( e \in G \) (identity), (2) a regular map \( m : G \times G \rightarrow G \) (multiplication), and (3) a regular map \( i : G \rightarrow G \) (inverse),
satisfying the group axioms: (i) \( m(e, a) = m(x, a) = a \) for all \( a \in G \),
(ii) \( m(m(a, b), c) = m(a, m(b, c)) \) for all \( a, b, c \in G \) and \( m(a, i(a)) = m(i(a), a) = e \) for all \( a \in G \). Recall that we defined a linear algebraic
group as a Zariski closed subgroup of \( \text{GL}_n(k) \) for some \( n \). If there
is no confusion, we will write \( a \cdot b \) or \( ab \) instead of \( m(a, b) \), and \( a^{-1} \)
instead of $i(a)$. From the problem on the last problem set follows that any linear algebraic group is an affine algebraic group.

A (left) regular action of $G$ on a Zariski closed set $X$ is a regular map $\mu : G \times X \to X$ which is also an action: (i) $\mu(e, x) = x$ for all $x \in X$ and (ii) $\mu(m(a,b), x) = \mu(a, \mu(b,x))$ for all $a, b \in G$ and all $x \in X$. If there is no confusion, we will write $a \cdot x$ instead of $\mu(a, x)$.

(a) Suppose that $\mu : G \times X \to X$ is an action of an affine algebraic group $G$ on the Zariski closed set $X$. For $f \in K[X]$ we define a function $g \cdot f$ on $X$ by $(g \cdot f)(x) = f(g^{-1} \cdot x)$. Show that $g \cdot f \in k[X]$, i.e., it is a regular function on $X$. Show that $G \times k[X] \to k[X]$ given by $(g,f) \mapsto (g \cdot f)$ defines a left action of $G$ on $k[X]$.

(b) Suppose that $W \subset k[X]$ is a finite dimensional subspace. Let $V := GW$ be the $k$-vector space spanned by all $g \cdot f$ with $g \in G$ and $f \in W$. Show that $V$ is a finite dimensional vector space as well. Show that the action of $G$ on $V$ is regular (so now $V$ is viewed as an affine space). Define also a regular action of $G$ on $V^*$, the dual space.

(c) Show that the inclusion $V \subset k[X]$ extends to a homomorphism $\psi : k[V^*] \to k[X]$. Assume $W$ were chosen such that it contains generators of $k[X]$. Then $\psi$ is surjective. Prove that $\psi = \phi^*$ corresponds to a regular map $\phi : X \to V^*$. (This is a closed immersion by definition.) Show that $\phi$ is respects the action: $\phi(g \cdot x) = g \cdot \phi(x)$ for all $g \in G$, $x \in X$. If $\phi$ respects the action, we will say that $\phi$ is $G$-equivariant.

(d) Note that any affine algebraic group $G$ acts on itself via $m$. Let $\phi : G \to V$ be an equivariant closed immersion into a vector space $V$ on which $G$ acts linearly and regularly. (As in the previous part, but with $V^*$ replaced by $V$.) Define $\lambda(g) \in \text{End}(V)$ by $\lambda(g)(v) = g \cdot v = \mu(g,v)$ for $v \in V$. Show that $\lambda$ induces an isomorphism of $G$ with a Zariski closed subset of $\text{GL}(V)$.

This shows that every affine algebraic group is in fact a linear algebraic group.