HOMEWORK 5 ON CHAPTER I
ALGEBRAIC GEOMETRY I
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(1) Suppose that R is a finite dimensional (not necessarily commutative) k-algebra R with 1. Let V be a finite dimensional left-A-module. Then V is called a *cyclic* module if there exists a $v \in V$ such that $Rv = V$. The module V is called *simple* if it does not contain any submodule W with $W \neq \{0\}$ and $W \neq V$.

Let k be an algebraically closed field. Let V be an n-dimensional k-vector space and let $\operatorname{End}(V)$ be the space of endomorphisms of V.

(a) For $A, B \in \operatorname{End}(V)$, let $k\langle A, B\rangle$ be the associative subalgebra of $\operatorname{End}(V)$ generated by A, B. Define

$$U_1 := \{(A, B) \in \operatorname{End}(V)^2 \mid V \text{ is a cyclic } k\langle A, B\rangle\text{-module}\}.$$

Prove that U_1 is open and dense. (Hint: define

$$U_1(v) = \{(A, B) \in \operatorname{End}(V)^2 \mid v \text{ is a cyclic vector for } V \text{ as a } k\langle A, B\rangle\text{-module}\}$$

and show that $U_1(v)$ is open for all $v \in V$.)

(b) Show that

$$U_2 := \{(A, B) \in \operatorname{End}(V)^2 \mid k\langle A, B\rangle = \operatorname{End}(V)\}$$

is open and dense. (Hint: Note that $k\langle A, B\rangle = \operatorname{End}(V)$ if and only if V^n is a cyclic $k\langle A, B\rangle$-module.)

(c) Define

$$U_3 := \{(A, B) \in \operatorname{End}(V)^2 \mid V \text{ is a simple } k\langle A, B\rangle\text{-module}\}.$$

Show that U_2 is open and dense as well. (Hint: Consider the varieties

$$Z_r = \{(A, B, W) \in \operatorname{End}(V)^2 \times \operatorname{Grass}(r, V) \mid AW \subseteq W \text{ and } BW \subseteq W\}$$

and project onto $\operatorname{End}(V)^2$.)

(d) * Prove that $U_2 = U_3$. (You may NOT use this though to solve (c).)

(2) Suppose that X is an affine variety and let G be a finite group acting on X regularly. Then G also acts coordinate ring $k[X]$. Let $k[X]^G$ be the subring of $k[X]$ of all invariant functions, so

$$k[X]^G = \{f \in K[X] \mid g \cdot f = f \text{ for all } g \in G\}.$$

We are going to prove that $k[X]^G$ is finitely generated. (The special case where the group order is not divisible by the characteristic of k can be found in the appendix.)

(a) Prove that $k[X]$ is integral over $k[X]^G$. (For every $f \in k[X]$, consider the polynomial

$$P(T) = \prod_{g \in G} (T - g \cdot f).)$$

(b) Prove that there is subalgebra R of $k[X]^G$ which is finitely generated over k, such that $k[X]$ is integral over R.

(c) Suppose that M is a finitely generated R module. Prove that any submodule of M is also finitely generated. (Reduce to the case where M is isomorphic to an ideal and use induction with respect to the number of generators of M. Use that R is Noetherian. You may also look this up in the literature.)

(d) Use (c) to prove that $k[X]^G$ is finitely generated.

(e) Now $k[X]^G$ is the coordinate of an affine variety. We denote this affine variety by X/G. The inclusion $k[X]^G \subseteq k[X]$ corresponds to a regular map $X \to X/G$ denoted by π. Prove that π is surjective and that all $\pi^{-1}(x), x \in X/G$ are exactly all G-orbits.

(3) Suppose that $\phi : X \to Y$ is a regular map between quasiprojective varieties. Prove that the following are equivalent:

(i) for every quasi-projective variety Z

$$\phi \times \mathrm{id} : X \times Z \to Y \times Z$$

is closed (i.e., maps closed sets to closed sets);

(ii) for every n the map

$$\phi \times \mathrm{id} : X \times \mathbb{A}^n \to Y \times \mathbb{A}^n$$

is closed.

(iii) There exists a projective variety Z and a closed immersion $\psi : X \to Z \times Y$ such that $\phi = p \circ \psi$ where $p : Z \times Y \to Y$ is the projection. (This is the "proper" property in the book, but you may also call such a regular map ϕ a projective map.)

(4) (Bonus) A regular map $\phi : X \to Y$ is called affine if for every $x \in Y$ there exists an open neighborhood U of x such that $\phi^{-1}(U)$ is open an affine as well.

(a) * Prove that if $\phi : X \to Y$ is affine, then for *every* $U \subseteq Y$ open and affine, $\phi^{-1}(U)$ is open and affine.

(b) * Prove that if $\phi : X \to Y$ is an affine, proper map between quasi-affine varieties, then ϕ is finite. (You are not allowed to use Theorem 7, §5.3 or any result beyond that.)