HOMEWORK 6 ON CHAPTER I
ALGEBRAIC GEOMETRY I
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(1) This problem contains a counterexample to the Corollary in §6.3. Suppose that the base field is $k = \mathbb{C}$. Let $X = \{(x, y, z) \in \mathbb{A}^3 \mid z \neq 0\}$ and $Y = \mathbb{A}^3$ and define $\phi : X \to Y$ by
$$\phi(x, y, z) = (x + z, xy, z^3 - z).$$
Note that X, Y are affine and irreducible.
  (a) Verify that ϕ is surjective.
  (b) Define
  $$Y_1 = \{y \in \mathbb{A}^3 \mid \dim \phi^{-1}(y) \geq 1\}.$$
  Determine Y_1 explicitly and show that Y_1 is not closed.
  (c) Find the error in the Proof of the Corollary in §6.3.

(2) (An alternative for the Corollary iin §6.3.) For a quasi projective variety X and $x \in X$ we define
$$\dim_x X = \max\{\dim X_i \mid x \in X_i\}.$$
where
$$X = X_1 \cup X_2 \cup \cdots \cup X_s$$
is a decomposition of X into irreducible varieties.
Let $\phi : X \to Y$ be a regular map between quasi-projective varieties. Define
$$X_l = \{x \in X \mid \dim_x \phi^{-1}(\phi(x)) \geq l\}$$
and
$$Y_l = \{y \in Y \mid \dim \phi^{-1}(y) \geq l\}.$$
  (a) Prove that X_l is closed for all l.
  (b) If ϕ is proper, show that Y_l is closed for all l.
  (c) Suppose that every fiber $\phi^{-1}(y)$ is irreducible, and that ϕ has a section, i.e., there exists a regular map $\psi : Y \to X$ such that the composition $\phi \circ \psi$ is the identity. Then Y_l is closed for all l.

Hence the Corollary in §6.3 is true in special situations. For example it is true if ϕ is proper. In particular it is true when X is a projective variety.

(3) Suppose $\mu : G \times X \to X$ is an action of an affine algebraic group G on a quasi-projective variety. We will write $g \cdot x$ instead of $\mu(g, x)$ for $g \in G$ and $x \in X$. The orbit of $x \in X$ is defined as
$$G \cdot x = \{g \cdot x \mid g \in G\}.$$
  (a) Prove that for every $x \in X$, the orbit $G \cdot x$ is open in its closure $\overline{G \cdot x}$. (So $G \cdot x$ can be seen as a quasi-projective variety. Hint: First prove that $G \cdot x$ contains a nonempty open subset U of $\overline{G \cdot x}$, then "smear U around".)

(b) Define $Z_l = \{x \in X \mid \dim(G \cdot x) \geq l\}$. Prove that Z_l is *open* for all l. (Hint: look at dimensions of *stabilizers*, and view stabilizers as fibers.)

(c) The additive group $G = \mathcal{G}_a = \mathbb{A}^1$ is an affine algebraic group with identity element $e = 0 \in \mathbb{A}^1$, multiplication $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ given by $(x, y) \mapsto x + y$ and the inverse $i : \mathbb{A}^1 \to \mathbb{A}^1$ given by $x \mapsto -x$. Prove that if $\mathcal{G}_a$ acts on an *affine* variety X, then all orbits are closed. (Hint: First prove that image of *any* regular map $\mathbb{A}^1 \to X$ is closed.)

(4) (a) Suppose that X is an irreducible projective variety. Prove that there exists a closed subset Y of codimension $\geq r$ which intersects with *every* closed subset Z of dimension $\geq r$. Moreover, given a closed subset W of X of dimension $\leq r - 1$, one can choose Y such that $Y \cap W = \emptyset$. (Hint: First do the case $X = \mathbb{P}^n$.)

(b) Explicitly give a closed curve in $\mathbb{P}^1 \times \mathbb{P}^1$ that intersects with every other closed curve in $\mathbb{P}^1 \times \mathbb{P}^1$.

(c) * Suppose that $\phi : X \times Y \to Z$ is a regular map, where X is irreducible quasiprojective, Y projective and Z is quasi-projective. Prove that $x \mapsto \dim \phi(\{x\} \times Y)$ is a constant function on X. (Hint: Without loss of generality we may assume that Z is projective as well. Choose $x_0 \in X$ with $r := \dim \phi(\{x_0\} \times Y)$ minimal. Choose $D \subseteq Z$ closed (of codimension $r + 1$) such that $D \cap \phi(\{x_0\} \times Y) = \emptyset$ and D intersects with every closed subvariety of Y of dimension $\geq r + 1$. Then we have

$$X = p(\phi^{-1}(D)) \cup \{x \in X \mid \dim \phi(\{x\} \times Y) \leq r\}.$$

where $p : X \times Y \to X$ is the projection onto the first factor.)

(d) An *algebraic group* is a quasi-projective variety G with a identity element $e \in G$ and regular maps $m : G \times G \to G$ (multiplication) and $i : G \to G$ (inverse) that satisfy the group axioms. (The same definition as for affine algebraic group, except that we do not assume that G is affine.) A regular action of G on a quasiprojective variety X is a regular map $\mu : G \times X \to X$ satisfying the axioms for an action. Suppose that G is a *projective* algebraic group acting regularly on an irreducible quasi-projective variety X. Show that all the orbits have the same dimension.

(e) Suppose that G is an *irreducible projective* algebraic group. Show that G is abelian. (Hint: Let G act on itself by conjugation and consider the dimensions of the orbits.) Irreducible projective algebraic groups are often called *abelian varieties*.