HOMEWORK 7 ON CHAPTER II
ALGEBRAIC GEOMETRY I
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(1) (Warm-up) Recall that the set of nonsingular points of a quasiprojective variety contains an open and dense subset. (In fact, it *is* open and dense itself.) Suppose that G is an algebraic group acting regularly on a quasiprojective variety X.
   (a) We have seen earlier that for every $x \in X$, then orbit $G \cdot x = \{g \cdot x \mid g \in G\}$ is locally closed set (i.e., open in its closure). In particular $G \cdot x$ can be seen as a quasiprojective variety. Show that $G \cdot x$ is also *nonsingular*.
   (b) Prove that G itself is a nonsingular (so algebraic groups are always nonsingular).

(2) Suppose that $\pi : Y \to X$ is a surjective map between quasi-projective varieties and that for every $x \in X$, $Y_x := \pi^{-1}(x)$ has the structure of an n-dimensional k-vector space. We say that $\pi : Y \to X$ is a vector bundle over X of rank n if it is "locally trivial" in the following sense: There exists an open covering $U_1, U_2, \dots, U_r$ of X and isomorphisms $\phi_i : \pi^{-1}(U_i) \to \mathbb{A}^n \times U_i$ such that (i) $p_i \circ \phi_i = \pi_i$ for all i, where $p_i : \mathbb{A}^n \times U_i \to U_i$ is the projection, and (ii) the isomorphism $Y_x = \pi^{-1}(x) \to \mathbb{A}^n \times \{x\}$ induced by ϕ_i is *linear* for all $x \in X$ (with respect to the vector space structures of Y_x and $\mathbb{A}^n$).

   Suppose that $X \subseteq \mathbb{A}^N$ is an irreducible, nonsingular, closed subset of dimension n. Let $\Theta \subseteq \mathbb{A}^N \times X$ and $\pi : \Theta \to X$ as in the beginning of Chapter II, §1.4. Show that $\pi : \Theta \to X$ is a vector bundle of rank $n = \dim X$. (With some more effort, a tangent bundle $\pi : \Theta \to X$ can be constructed for *any* irreducible nonsingular *quasi-projective* variety X.)

(3) Suppose that X is an irreducible quasiprojective variety of dimension n which is smooth at $x \in X$. If $f_1, f_2, \dots, f_n \in k[X]$, then we define the intersection multiplicity $\operatorname{mult}_x(f_1, \dots, f_n)$ at x as the dimension of

$$\mathcal{O}_x/(f_1, \dots, f_n)$$

as a k-vector space, where $\mathcal{O}_x$ is the local ring at $x \in X$, and $(f_1, \dots, f_n)$ is the ideal in $\mathcal{O}_x$ generated by (the images of) $f_1, \dots, f_n$. (This intersection multiplicity may be infinite.)
   (a) Show that $\operatorname{mult}_x(f_1, \dots, f_n) = 0$ if and only if $f_i(x) \neq 0$ for some i.
   (b) Show that $1 \leq \operatorname{mult}_x(f_1, \dots, f_n) < \infty$ if and only if there exists an open neighborhood U of x such that x is the only solution of $f_1 = f_2 = \cdots = f_n = 0$ in U.
   (c) Suppose that $X = \mathbb{A}^2$, and $f, g \in k[X] \cong k[x, y]$ with f irreducible and g square-free. Let Y and Z be the curves $Y = \{f = 0\}$ and $Z = \{g = 0\}$ respectively. Prove that $\operatorname{mult}_{(0,0)}(f, g)$ defined here is equal to the intersection multplicity of

Y and Z at $(0,0)$ defined before (see bottom of page 14 in §1.5 of Shafarevich) if Y is smooth at $(0,0)$.

(4) (a) Suppose that R is a Noetherian domain and I is an ideal with $I \neq R$, prove that

$$\bigcap_{r=0}^{\infty} I^r = (0).$$

(Hint: use the Nakayama Lemma.)

(b) Suppose that $\mathcal{O}$ is a Noetherian local ring (not necessarily a domain) with maximal ideal $\mathfrak{m}$. Prove that

$$\bigcap_{r=0}^{\infty} \mathfrak{m}^r = 0.$$

(Note that we used this fact in the lecture to construct the tangent cone.)

(5) (Bonus)

(a) * Suppose that R is a ring that is finitely generated field k. Suppose that W is a subspace of R (as a k-vector space) containing k which generates R a a ring over k. Let W^r be the subspace of R spanned by all possible products of r elements in W. We define $d(r)$ as the dimension of W^r (as a k-vector space). Prove that

$$\text{GK-dim}\, R := \limsup_{r \to \infty} \frac{\log d(r)}{\log r}$$

does not depend on the choice of W.

(b) * Show that if X is an affine variety of dimension n, then GK-dim $k[X]$ is well-defined and equal to n. (Hint: Reduce to the case that X is irreducible. Use a Noether Normalization, Chapter I, §5.4, Theorem 10.)

(c) * Show that GK-dim R is always finite if R is finitely generated over a k. Show that $\text{GK-dim}\, R = \text{GK-dim}\, R/\sqrt{0}$.

(d) * Suppose that X is a quasiprojective variety and that $x \in X$. The coordinate ring of the tangent cone T_x is $R/\sqrt{0}$ where

$$R := \mathrm{gr}_{\mathfrak{m}_x} \mathcal{O}_x = \bigoplus_{i=0}^{\infty} \mathfrak{m}_x^i / \mathfrak{m}_x^{i+1},$$

where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_x$ of X at x. Show that the dimension of the tangent cone T_x is equal to $\dim_x X$.

Remark: Here GK-dim stands for *Gelfand-Kirillov dimension*. This notion generalizes to noncommutative rings. We do not even have to assume that R is finitely generated: If R is not finitely generated over k then we can define $\text{GK-dim}\, R = \sup_S \text{GK-dim}\, S$ where the supremum is taken over all finitely generated k-subalgebras of R.