LECTURE 1: QUIVER REPRESENTATIONS AND LINEAR ALGEBRA

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1. FIRST DEFINITIONS

Definition 1. A **quiver** Q is a directed graph $Q = (Q_0, Q_1)$ where Q_0 is the set of vertices and Q_1 is the set of arrows. The maps $h : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ map each arrow to its head and its tail respectively.

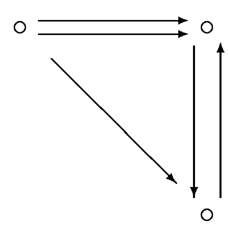
To avoid any confusion with Graph Theory which is quite different from the theory in this course, it is common to speak of quivers instead of directed graph (although it is the same thing). **Throughout the course, all quivers are finite, i.e., finitely many vertices and arrows.** Typically, we would take for Q_0 a set of numbers and for Q_1 a set of letters.

Example 1. For the quiver

$$1 \xrightarrow{\quad a \quad} 2 \xrightarrow{\quad b \quad} 3$$

we have $Q_0 = \{1, 2, 3\}$ and $Q_1 = \{a, b\}$. We also get $ta = 1$, $ha = tb = 2$ and $hb = 3$.

Example 2. A quiver may have several arrows in the same or in opposite direction, and it may also have loops. For example:



We now fix a base field K. **For simplicity, K is assumed to be algebraically closed and of characteristic 0 throughout the course.**

Roughly speaking, if you attach to each vertex a vector space and to each arrow a linear map, then this is what we call a quiver representation. Let us give a more precise definition.

Definition 2. Suppose that Q is a quiver. A **representation V of Q** is a set

$$\{V(x) \mid x \in Q_0\}$$

of finite dimensional K-vector spaces together with a set

$$\{V(a) : V(ta) \to V(ha) \mid a \in Q_1\}$$

of K-linear maps.

Example 3. Consider the quiver

$$1 \xrightarrow{a} 2.$$

Suppose that m and n are positive integers and M is an $m \times n$ matrix. Then we can define a representation V_M by $V_M(1) = K^m$, $V_M(2) = K^n$ and $V_M(a) = M$.

Definition 3. Suppose that V and W are representations of a quiver Q. A **morphism** $\phi : V \to W$ is a collection of K-linear maps

$$\{\phi(x) : V(x) \to W(x) \mid x \in Q_0\}$$

such that for every arrow $a \in Q_1$ the diagram

$$\begin{array}{ccc} V(ta) & \xrightarrow{V(a)} & V(ha) \\ {\scriptstyle \phi(ta)}\downarrow & & \downarrow{\scriptstyle \phi(ha)} \\ W(ta) & \xrightarrow[W(a)]{} & W(ha) \end{array}$$

commutes, i.e., $\phi(ha)V(a) = W(a)\phi(ta)$. If moreover, $\phi(x)$ is invertible for every $x \in Q_0$, then ϕ is called an **isomorphism**.

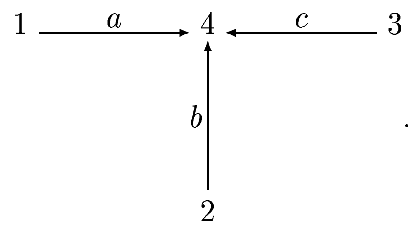
Example 4. Consider again the quiver

$$1 \xrightarrow{a} 2.$$

Suppose that A and B are two $m \times n$ matrices. When are the quiver representations V_A and V_B isomorphic? According to the definition, we must have invertible linear maps $\phi(1) : K^m \to K^m$ and $\phi(2) : K^n \to K^n$ such that $\phi(2)A = B\phi(1)$ or equivalently $\phi(2)A\phi(1)^{-1} = B$. In other words, V_A and V_B are isomorphic quiver representations if and only if B can be obtained from A by a change of basis in K^m and a change of basis in K^n.

Example 5. Consider the quiver Q with one vertex called 1, and one loop called a. For every positive integer n and every square $n \times n$ matrix M we can define a quiver representation V_M with $V_M(1) = K^n$ and $V_M(a) = M$. If A, B are two $n \times n$ matrices, then V_A and V_B are isomorphic if and only if there exists a linear map $\phi(1) : K^n \to K^n$ such that $\phi(1)A = B\phi(1)$, or equivalently $\phi(1)A\phi(1)^{-1} = B$. In other words, V_A and V_B are isomorphic quiver representations if and only if the matrices A and B are conjugate. Note also that any representation V is isomorphic to V_M for some matrix M (just choose a basis of $V(1)$, then the map $V(a)$ is given by a square matrix M).

Example 6. Let Q be the quiver



Let us for a moment only look at representations V with $V(a)$, $V(b)$, $V(c)$ injective. We can think of such a representation as a vector space $V(4) \cong K^n$ with three subspaces $V_1 := V(a)V(1)$, $V_2 := V(b)V(2)$ and $V_3 := V(c)V(3)$. On the other hand, if V_1, V_2, V_3 are subspaces of K^n, we can make a quiver representation by $V(i) := V_i$ for $i = 1, 2, 3$, $V(4) = K^n$, and $V(a)$, $V(b)$ and $V(c)$ are the inclusion maps $V_1 \hookrightarrow K^n$, $V_2 \hookrightarrow K^n$ and $V_3 \hookrightarrow K^n$ respectively. Suppose that W_1, W_2, W_3 are also 3 subspaces of K^n. In a similar fashion, we can make a quiver representation W out of this. It is an exercise (Exercise 2) to show that V and W are isomorphic if and only if there exists a invertible $n \times n$ matrix A such that $AV_i = W_i$ for $i = 1, 2, 3$.

2. IRREDUCIBLE AND INDECOMPOSABLE REPRESENTATIONS

Definition 4. Suppose that V and W are representations of Q. The representation W is called a **subrepresentation** of V if

1. For every $x \in Q_0$, $W(x)$ is a subspace of $V(x)$ and
2. for every $a \in Q_1$, the restriction of $V(a) : V(ta) \to V(ha)$ to $W(ta)$ is equal to $W(a) : W(ta) \to W(ha)$.

Note that every quiver has a a representation Z with $Z(x) = 0$ for all $x \in Q$ (and $Z(a) = 0$ for all $a \in Q_1$). This representation is called the **zero representation** or the **trivial representation**.

Definition 5. A nonzero representation V is called **irreducible** or **simple** if the only subrepresentations of V are the zero representation and V itself.

There are some obvious irreducible representations which can be defined as follows. For every $x \in Q_0$, define a representation E_x by $E_x(y) = K$ if $y = x$ and $E_x(y) = 0$ if $y \in Q_0 \setminus \{x\}$. For every arrow a, $E_x(a)$ is the zero map.

Example 7. Let Q be the quiver

$$1 \underset{b}{\overset{a}{\rightleftarrows}} 2$$

For every $\lambda \in K \setminus \{0\}$ we have an irreducible representation

$$K \underset{1}{\overset{\lambda}{\rightleftarrows}} K.$$

Other irreducible representations are

$$E_1 = K \rightleftarrows 0 \text{ and } E_2 = 0 \rightleftarrows K.$$

(One can show that these are the only irreducible representations of the quiver Q.)

Definition 6. If V and W are representations of a quiver Q, then we can define the **direct sum representation** $V \oplus W$ by

$$(V \oplus W)(x) = V(x) \oplus W(x)$$

for every $x \in Q_0$, and $(V \oplus W)(a) : V(ta) \oplus W(ta) \to V(ha) \oplus W(ha)$ is defined by the matrix

$$\begin{pmatrix} V(a) & 0 \\ 0 & W(a) \end{pmatrix}.$$

Definition 7. A representation V is called **decomposable** if V is isomorphic to a direct sum $W \oplus X$ where W and X are nonzero representations. A nonzero representation is called **indecomposable** if it is not decomposable.

Since all the vector spaces are finite dimensional, it is easy to see that all that every representation can be written as a finite sum of indecomposable representations. In fact, this can be done so uniquely (up to isomorphism and permutation) by a theorem of Krull and Schmid (which will be discussed later).

Example 8. Let Q be the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} n$$

This quiver we will sometimes call the "equioriented A_n". For integers j, k with $1 \leq j < k \leq n$ we define an indecomposable representation $I_{j,k}$ by

$$0 \to \cdots \to 0 \to \underset{j}{K} \to K \to \cdots \to \underset{k}{K} \to 0 \to \cdots \to 0.$$

(All nontrivial maps are identity.) In fact, one can prove that these are all the indecomposable representations.

Example 9. Let Q be the quiver with one vertex (1) and one loop (a). As before, for an $n \times n$ matrix A we define a representation V_A of Q with $V_A(1) = K^n$ and $V_A(a) = A$. If B and C are $k \times k$ and $n - k \times n - k$ matrices respectively, then $V_A \cong V_B \oplus V_C$ if and only if A is conjugated to the block matrix

$$\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}.$$

For $\lambda \in K$, let $J_{\lambda,n}$ be the $n \times n$ matrix

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}.$$

(a Jordan-block!) Let $V_{\lambda,n} := V_{J_{\lambda,n}}$ be the corresponding representation of Q. One can show that $V_{\lambda,n}$ is indecomposable (exercise). Every matrix is conjugated to a block matrix with Jordan blocks on the diagonal and 0 outside. Therefore, for every representation V of Q, we have

$$V \cong V_{\lambda_1,k_1} \oplus V_{\lambda_2,k_2} \oplus \cdots \oplus V_{\lambda_r,k_r}$$

for some $\lambda_1, \dots, \lambda_r \in K$ and some positive integers $k_1, \dots, k_r$. This also shows that the $V_{\lambda,n}$ are *all* indecomposable representations.

3. About the Course

The goal of this course is to cover the following topics:

1. Discussion of quivers with *finitely many* indecomposable representations. These quivers are called **of finite representation type**. There is a beautiful theorem by Gabriel that a quiver is of finite representation type if and only if the underlying diagram (the undirected graph obtained by forgetting the orientation of all arrows) is a Dynkin diagram of type A, D or E. Gabriel also proved a correspondence between possitive roots of root systems and indecomposable representations. We will discuss root systems, simple Lie algebras and its connection to quiver representations. We may also discuss generalizations to quivers which are not of finite representation type (so-called tame and wild quivers).
2. We will study the so-called path algebra of a quiver. Quiver representations can be viewed within the framework of representations of finite dimensional associative algebras (not necessarily commutative). The representation theory of finite dimensional algebras has developped a lot in the last decades. We discuss one of the basic tools in this theory, Auslander-Reiten sequences and apply it to quivers (of finite representation type).
3. Following Kac, Schofield and others, we will develop some theory about "general representations". For example, if we fix the dimensions of the vector spaces $V(x)$ for all $x \in Q_0$, and we choose the linear maps $V(a)$ in general position, what can be said about the decomposition of V into indecomposable representations? This and other related questions will be studied.
4. We will discuss invariant theory and its application to quiver representations. There is also a geometric way of looking at the invariant theory for quivers.

Roughly speaking, it enables us to view the set of isomorphism classes of quiver representations as an algebraic variety.

5. We will discuss the representation theory of the general linear group and its relation to the invariant theory for quiver representations. The ultimate goal is to use the reprentation theory of quivers to obtain results for the representation theory of the general linear group. In particular we hope to prove the so-called saturation problem for Littlewood-Richardson coefficients using quiver reprsentations.

4. EXERCISES

Exercise 1. An **oriented cycle** is a sequence $a_1, a_2, \dots, a_l$ $(l \geq 1)$ with $ha_i = ta_{i+1}$ and $ha_l = ta_1$. Suppose that Q is a quiver with n vertices for which there does not exist an oriented cycle. Show that you can find a bijection

$$\psi : Q_0 \to \{1, 2, \dots, n\}$$

(i.e., a labeling of the vertices by $1, 2, \dots, n$) such that $\psi(ta) < \psi(ha)$ for every $a \in Q_1$. In other words, without loss of generality we may assume that $Q_0 = \{1, 2, \dots, n\}$ and $ta < ha$ (as comparison of integers).

Exercise 2. Show that in Example 6, V and W are isomorphic if and only if there exists an invertible linear map $A : K^n \to K^n$ such that $AV_i = W_i$ for $i = 1, 2, 3$.

Exercise 3. Let Q be the quiver

$$1 \xrightarrow{\quad a \quad} 2.$$

(a). Show that the only indecomposable representations are

$$E_1 = K \longrightarrow 0, E_2 = 0 \longrightarrow K, I = K \xrightarrow{\quad 1 \quad} K.$$

(b). Suppose that V_A is an representation of Q with $V_A(1) = K^n$, $V_A(2) = K^m$ and $V_A(a) = A : K^n \to K^m$. Show that $V \cong E_1^{d_1} \oplus E_2^{d_2} \oplus I^r$ with d_1 the dimension of the kernel of A, d_2 the dimension of the cokernel of A and r the rank of A.

(c). Show that I is not irreducible.

Exercise 4. Consider the quiver with one vertex (1) and one arrow (a). Let us drop for a moment the assumption that the base field is algebraically closed and let us take $K = \mathbb{R}$. Let V be the representation with $V(1) = \mathbb{R}^2$ and

$$V(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(a). Show that V is indecomposable and even irreducible as a representation of Q over the base field $\mathbb{R}$.

(b). However, if we "complexify", i.e., take the base field $K = \mathbb{C}$, $V(1) = \mathbb{C}^2$ and $V(a)$ given by the same matrix as before, then V is decomposable.

Exercise 5. Suppose that Q is a quiver without oriented cycles. We may assume that $Q_0 = \{1, 2, \ldots, n\}$ and $ta < ha$ for all $a \in Q_1$ (see Exercise 1). Show that $E_1, E_2, \ldots, E_n$ are the only irreducible representations.

Exercise 6. Let Q be the quiver with one vertex (labeled 1) and one loop (labeled a). Show that a representation V is irreducible if and only if $V(1)$ is one-dimensional. (Use that every matrix over an algebraically closed field has an eigenvector.)

Exercise 7. Let Q be the quiver with one vertex (labeled 1) and two loops (a and b). Let V_n be the representation with $V_n(1) = K^n$ and $V_n(a)$ and $V_n(b)$ defined by the matrices

$$\begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ 0 & & & 1 & 0 \end{pmatrix}$$

respectively. Show that V_n is irreducible.

Exercise 8. Show that the $I_{j,k}$ in Example 8 are the only indecomposable representations. For example, one could proceed as follows: Assume that V is an indecomposable representation. Let k be the largest integer such that $V(k) \neq 0$. Let j be the smallest integer such that the map $V(a_{k-1})V(a_{k-2})\cdots V(a_j)$ is not the zero map. We can find a $v_j \in V(j)$ such that

$$V(a_{k-1})V(a_{k-2})\cdots V(a_j)v_j \neq 0.$$

For $i = j+1, \ldots, k$ define $v_i = V(a_{i-1})V(a_{i-2})\cdots V(a_j)v_j$. Notice that $v_k \neq 0$. Define a subrepresentation V_1 of V by $V_1(i) = 0$ if $i < j$ or $i > k$ and $V_1(i) = Kv_i$ if $i = j, \ldots, k$. Show that V_1 is a well-defined subrepresentation and that it is isomorphic to $I_{j,k}$. Define another subrepresentation V_2 of V as follows. Choose a complement C_k to $v_k \in V(k)$, i.e., $V(k) = C_k \oplus Kv_k$. Define $C_i = 0$ for $i > k$ and

$$C_i = (V(a_{k-1})V(a_{k-2})\cdots V(a_i))^{-1}(C_k)$$

for $i < k$. Show that we can define a subrepresentation V_2 of V by $V_2(i) = C_i$ for all i. Show that $V(i) = V_1(i) \oplus V_2(i)$ for all i. Since V is indecomposable we must have $V_2 = 0$ and $V = V_1 \cong I_{j,k}$.

Exercise 9. Show that the representations $V_{\lambda,n}$ defined in Example 9 are indecomposable. Note that for $n \geq 2$, these representations are not irreducible.