LECTURE 10: SOME MORE RESULTS ON SEMI-INVARIANTS

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In the previous lecture we have proven the following theorem.

Theorem 1 (Derksen-Weyman). *Suppose that Q is a quiver without oriented cycles, β is a dimension vector and σ is a weight. Then $\mathrm{SI}(Q,\beta)_\sigma$ is spanned by c^V's where $V \in \mathrm{Rep}(Q,\alpha)$ and $\sigma = \langle \alpha, \cdot \rangle$.*

We now discuss some consequences of this theorem. We also will give an alternative proof of the theorem, based on another theorem of Derksen-Schofield-Weyman (which will not be proven here).

Corollary 1. (Reciprocity Property). *Let α, β be two dimension vectors for the quiver Q. Assume that $\langle \alpha, \beta \rangle = 0$. Then*

$$\dim_K \mathrm{SI}(Q,\beta)_{\langle \alpha, \cdot \rangle} = \dim_K \mathrm{SI}(Q,\alpha)_{-\langle \cdot, \beta \rangle}.$$

Proof. Let $V_1, \dots, V_s$ be the modules of dimension α such that $c^{V_1}, \dots c^{V_s}$ form a basis of $\mathrm{SI}(Q,\beta)_{\langle \alpha, \cdot \rangle}$. These are linearly independent polynomials on $\mathrm{Rep}(Q,\beta)$ so there exist s representations $W_1, \dots, W_s$ in $\mathrm{Rep}(Q,\beta)$ such that $\det(c^{V_i}(W_j))_{1 \leq i,j \leq s}$ is not zero. But $c^{V_i}(W_j) = c_{W_j}(V_i)$ and this means that the semi-invariants $c_{W_1}, \dots, c_{W_s}$ are linearly independent. This proves that

$$\dim_K \mathrm{SI}(Q,\beta)_{\langle \alpha, \cdot \rangle} \leq \dim_K \mathrm{SI}(Q,\alpha)_{-\langle \cdot, \beta \rangle}.$$

The other inequality is proven in exactly the same way. □

Definition 1. We define

$$\alpha \circ \beta := \dim \mathrm{SI}(Q,\beta)_{\langle \alpha, \cdot \rangle} = \dim \mathrm{SI}(Q,\alpha)_{-\langle \cdot, \beta \rangle}.$$

We now give a sketch of an alternative proof of Theorem 1. For $V \in \mathrm{Rep}(Q, \alpha+\beta)$, let $r_\alpha(V) \in \mathbb{N} \cup \{\infty\}$ be the number of α-dimensional subrepresentations. It can be shown that the function r_α is constant k on some dense open subset $U \subseteq \mathrm{Rep}(Q, \alpha+\beta)$ for some $k \in \mathbb{N} \cup \{\infty\}$. We say that a general representation of dimension $\alpha + \beta$ has k subrepresentations.

Theorem 2 (Derksen-Schofield-Weyman). *A general representation of dimension $\alpha + \beta$ has exactly $\alpha \circ \beta$ α-dimensional subrepresentations.*

Proof. We will not prove this here. The proof is based on comparing the computation of the dimension $\mathrm{SI}(Q,\beta)_{\langle \alpha, \cdot \rangle}$ using the technique which we applied to the triple flag quiver to the computation of the number of subrepresentations of a general representation of dimension $\alpha + \beta$ using Schubert calculus. Both (complicated) computations are exactly analogous and therefore give the same result. □

We also need the following Lemma which we will prove some time in the future.

Lemma 1. *Suppose* $\langle \alpha, \beta \rangle = 0$, *and let* $S \subset \mathrm{Rep}(Q, \alpha + \beta)$ *be the set of all* $V \in \mathrm{Rep}(Q, \alpha + \beta)$ *for which there exists an exact sequence*

$$0 \to W \to V \to Z \to 0$$

with W *of dimension* α, Z *of dimension* β *and* $\mathrm{Hom}_Q(W, Z) \neq 0$. *Then* $\overline{S}$ *is a proper subset of* $\mathrm{Rep}(Q, \alpha + \beta)$.

Proof of Theorem 1. Let V be a general representation of dimension $\alpha + \beta$ and let $V_1, \dots, V_m$ be its α-dimensional subrepresentations with $m = \alpha \circ \beta$. We have exact sequences

$$0 \to V_i \to V \to W_i \to 0$$

for all i with W_i of dimension β. Since V is general, we may assume that it lies outside $\overline{S}$ and therefore $\mathrm{Hom}_Q(V_i, W_i) = 0$ and in particular $c^{V_i}(W_i) \neq 0$.

Let $i \neq j$ and consider

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & V_i & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & V_j & \to & V & \to & W_j & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & W_i & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

Suppose that the composition $V_i \to V \to W_j$ is equal to 0. Then $V_i \to V$ lifts to an injective morphism $V_i \to V_j$ and since V_i and V_j are both α-dimensional we get

$V_i \cong V_j$ but we assumed that the V_i's are distinct. Therefore $\mathrm{Hom}_Q(V_i, W_j) \neq 0$ which shows that $c^{V_i}(W_j) = 0$ if $j \neq i$.

It now clearly follows that the matrix $(c^{V_i}(W_j))_{i,j=1}^m$ is invertible, so $c^{V_1}, \dots, c^{V_m} \in \mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle}$ are linearly independent. Moreover $m = \alpha \circ \beta = \dim \mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle}$, so $c^{V_1}, \dots, c^{V_m}$ is a basis of $\mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle}$. □

1. The Theorems of Domokos-Zubkov and LeBruyn-Procesi

We first derive a recent result of Domokos and Zubkov who described spanning semi-invariants for arbitrary quivers (possibly with oriented cycles). Then we discuss a theorem of LeBruyn and Procesi about generators the ring of invariants for arbitrary quivers. This theorem (proven in 1990, before the theorems of Domokos-Zubkov and Derksen-Weyman) can easily be deduced from the Domokos-Zubkov description.

Theorem 3 (Domokos-Zubkov). *Let Q be an arbitrary quiver and let β be a dimension vector. The ring $\mathrm{SI}(Q,\beta)$ is spanned by semiinvariants of the form $W \in \mathrm{Rep}(Q,\beta) \mapsto \det(g)$ where*

$$g : \bigoplus_{i=1}^{n} W(x_i) \to \bigoplus_{j=1}^{m} W(y_j)$$

with $x_1, \dots, x_n, y_1, \dots, y_m \in Q_0$ such that $\sum_i \beta(x_i) = \sum_j \beta(y_j)$ and where

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ g_{2,1} & g_{2,2} & & g_{2,n} \\ \vdots & & \ddots & \vdots \\ g_{m,1} & g_{m,2} & \cdots & g_{m,n} \end{pmatrix}$$

with $g_{i,j}$ a linear combination of $W(p)$ where p is a path satisfying $tp = x_i$ and $hp = y_j$.

Proof. If Q has no oriented cycles then the Schofield semi-invariants c^V are actually of this type as described in the Theorem. The semi-invariant c^V is the determinant of the map

(1)
$$g : \bigoplus_{x \in Q_0} \mathrm{Hom}(V(x), W(x)) = \bigoplus_{x \in Q_0} W(x)^{\alpha(x)} \to \bigoplus_{a \in Q_1} \mathrm{Hom}(V(ta), W(ha)) = \bigoplus_{a \in Q_1} W(ha)^{\alpha(ta)},$$

where α is the dimension vector of V. Also, all blocks in the matrix of g are equal to 0, identity or $W(a)$ for some arrow a. This proves the case if Q has no oriented cycles.

To prove the general case, we use the double quiver construction of Schofield. We define a new quiver $\widehat{Q} = (\widehat{Q}_0, \widehat{Q}_1)$. We define $\widehat{Q}_0 = Q_0 \times \{0,1\}$, so for every vertex $x \in Q_0$ we get two vertices $x_0, x_1 \in \widehat{Q}_0$. Then we define

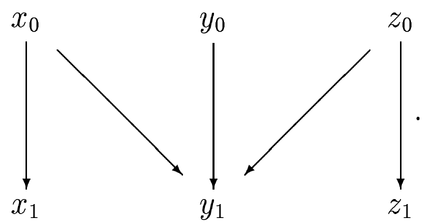
$$\widehat{Q}_1 = \{\widehat{a} \mid a \in Q_1\} \cup \{c_x \mid x \in Q_0\}.$$

We define $t\widehat{a} = (ta)_0$, $h\widehat{a} = (ha)_1$, $tc_x = x_0$ and $hc_x = 1$.

Example 1. Suppose that Q is the quiver

$$x \longrightarrow y \longleftarrow z$$

then $\widehat{Q}$ is the quiver



Note that whatever Q is, $\widehat{Q}$ will be a quiver without oriented cycles (in fact all paths have length 0 or 1). We define the dimension vector $\widehat{\beta}$ by $\widehat{\beta}(x_i) = \beta(x)$ for $i = 0, 1$ and $x \in Q_0$. Let us define $U(\beta) \subseteq \mathrm{Rep}(\widehat{Q}, \widehat{\beta})$ to be the set of all representations W with $W(c_x)$ invertible for all $x \in Q_0$. We have an isomorphism

$$U(\beta) \to \mathrm{Rep}(Q, \beta) \times \mathrm{GL}(\beta) \tag{2}$$

defined by

$$W \mapsto \{W(c_{ha})^{-1} W(a) \mid a \in Q_1\} \times \{W(x) \mid x \in Q_0\}.$$

Note that $\mathrm{GL}(\widehat{\beta}) \cong \mathrm{GL}(\beta) \times \mathrm{GL}(\beta)$ acts on $U(\beta)$ and there is also an action of $\mathrm{GL}(\beta) \times \mathrm{GL}(\beta)$ on $\mathrm{Rep}(Q, \beta) \times \mathrm{GL}(\beta)$ (one copy acts on $\mathrm{Rep}(Q, \beta)$ and on $\mathrm{GL}(\beta)$ on the right and the other copy acts on $\mathrm{GL}(\beta)$ by left multiplication). The isomorphism in (2) is $\mathrm{GL}(\beta) \times \mathrm{GL}(\beta)$-equivariant (i.e., it respects the action). In particular, is we take $\mathrm{SL}(\beta) \times \mathrm{SL}(\beta)$-invariants on both sides in (2) then we get

$$K[U_\beta]^{\mathrm{SL}(\beta) \times \mathrm{SL}(\beta)} = K[\mathrm{Rep}(Q, \beta) \times \mathrm{GL}(\beta)]^{\mathrm{SL}(\beta) \times \mathrm{SL}(\beta)}. \tag{3}$$

If we take $t_x : \mathrm{GL}(\beta(x)) \to K^\star$ to be the determinant function then $K[\mathrm{GL}(\beta)]^{\mathrm{SL}(\beta)} = K[t_x, t_x^{-1}]_{x \in Q_0}$ (here $\mathrm{SL}(\beta)$ acts on the left on $\mathrm{GL}(\beta)$. This shows that

$$K[\mathrm{Rep}(Q, \beta) \times \mathrm{GL}(\beta)]^{\mathrm{SL}(\beta) \times \mathrm{SL}(\beta)} = (K[\mathrm{Rep}(Q, \beta)][t_x, t_x^{-1}]_{x \in Q_0})^{\mathrm{SL}(\beta)}$$

where $\mathrm{SL}(\beta)$ acts trivially on the variables t_x, so

$$K[\mathrm{Rep}(Q, \beta) \times \mathrm{GL}(\beta)]^{\mathrm{SL}(\beta) \times \mathrm{SL}(\beta)} = K[\mathrm{Rep}(Q, \beta)]^{\mathrm{SL}(\beta)}[t_x, t_x^{-1}]_{x \in Q_0} = \mathrm{SI}(Q, \beta)^{\mathrm{SL}(\beta)}[t_x, t_x^{-1}]_{x \in Q_0}.$$

Let $D_x : \mathrm{Rep}(\widehat{Q}, \widehat{\beta}) \to K$ be the $\mathrm{SL}(\beta) \times \mathrm{SL}(\beta)$-invariant function $W \mapsto \det(W(c_x))$. The coordinate ring $K[U(\beta)]$ is the localization of $K[\mathrm{Rep}(\widehat{Q}, \widehat{\beta})]$

with respect to all D_x, $x \in Q_0$. Note that localization with respect to an invariant commutes by taking invariants, i.e., if $D = \prod_{x \in Q_0} D_x$, then

$$K[U(\beta)]^{\mathrm{SL}(\widehat{\beta})} = (K[\mathrm{Rep}(\widehat{Q}, \widehat{\beta})]_D)^{\mathrm{SL}(\widehat{\beta})} = (K[\mathrm{Rep}(\widehat{Q}, \widehat{\beta})]^{\mathrm{SL}(\widehat{\beta})}) = \mathrm{SI}(\widehat{Q}, \widehat{\beta})[D_x^{-1}]_{x \in Q_0}.$$

So we conclude that (3) leads to

$$\mathrm{SI}(\widehat{Q}, \widehat{\beta})[D_x^{-1}]_{x \in Q_0} \cong \mathrm{SI}(Q, \beta)[t_x, t_x^{-1}]_{x \in Q_0}. \tag{4}$$

Under this isomorphism, D_x maps to t_x. Consider the diagram

$$\begin{array}{ccc} \mathrm{Rep}(Q,\beta) & & \\ \downarrow & \searrow^{\psi} & \\ \mathrm{Rep}(Q,\beta) \times \mathrm{GL}(\beta) & \cong & U(\beta) \end{array}$$

where $\mathrm{Rep}(Q, \beta) \to \mathrm{Rep}(Q, \beta) \times \mathrm{GL}(\beta)$ is defined by $W \mapsto (W, \mathrm{id})$. Now this induces

$$\begin{array}{ccc} K[\mathrm{Rep}(Q,\beta)] & & \\ \uparrow & \nwarrow^{\psi^\star} & \\ K[\mathrm{Rep}(Q,\beta) \times \mathrm{GL}(\beta)] & \cong & K[U(\beta)] \end{array}$$

So we get a surjective ring homomorphism $\psi^\star : K[U(\beta)] \to K[\mathrm{Rep}(Q, \beta)]$ and if we take $\mathrm{SL}(\beta) \times \mathrm{SL}(\beta)$ invariants then this induces a surjective map of invariant rings

$$\mathrm{SI}(\widehat{Q}, \widehat{\beta})[D_x^{-1}]_{x \in Q_0} \cong \mathrm{SI}(Q, \beta)[t_x, t_x^{-1}]_{x \in Q_0} \to \mathrm{SI}(Q, \beta),$$

where t_x and D_x map to 1. If $W \in \mathrm{Rep}(Q, \beta)$, then $\psi(W)(\widehat{a}) = W(a)$ for all $a \in Q_1$ and $\psi(W)(c_x) = \mathrm{id}$ for all $x \in Q_0$

We know that $\mathrm{SI}(\widehat{Q}, \widehat{\beta})$ is spanned by invariants of the form $W \in \mathrm{Rep}(\widehat{Q}, \widehat{\beta}) \mapsto \det(g)$ where $g = (g_{i,j})$ and each $g_{i,j}$ is a linear combination of $W(p)$'s with p a path in $\widehat{Q}$. It follows that $\mathrm{SI}(Q, \beta)$ is generated by $W \in \mathrm{Rep}(Q, \beta) \mapsto \det(g)$ where $g = (g_{i,j})$ and each $g_{i,j}$ is a linear combination of $\psi(W)(p)$'s with p a path in $\widehat{Q}$. But in fact $\psi(W)(p) = W(q)$ for some path q in Q. (Note that either p has length zero or length 1, and if p has length one then either $p = \widehat{a}$ for some $a \in Q_1$ or $p = c_x$ for some $x \in Q_0$).) We have proven the theorem for the quiver Q. $\square$

For an abitrary quiver, let $\mathrm{Loop}(Q)$ be the set of loops, i.e., the set of all paths p with $hp = tp$.

Theorem 4 (LeBruyn-Procesi). *The invariant ring* $\mathrm{I}(Q, \beta)$ *is generated by* $\{\mathrm{Trace}(W(p)) \mid p \in \mathrm{Loop}(Q)\}$ *(*Trace *is the trace).*

Proof. Note that $\mathrm{I}(Q,\beta)=\mathrm{SI}(Q,\beta)_0$. In the Domokos-Zubkov Theorem, we only get semi-invariants of weight 0 if the sequences $x_1,\dots,x_n$ and $y_1,\dots,y_m$ are the same after rearranging. So $\mathrm{I}(Q,\beta)$ is generated by invariants of the form $\det(g)$ where

$$g:\bigoplus_{i=1}^{n} W(x_i)\to\bigoplus_{i=1}^{n} W(x_i)$$

and $g_{i,j}$ is a linear combination of $W(p)$'s with $tp=i$ and $hp=j$. Now by the Lemma below we have that $\det(g)$ is a polynomial in $\mathrm{Trace}(g^k)$, $k=1,2,\dots$ (we assume that $\mathrm{char}(K)=0$). If we write

$$g^k=\begin{pmatrix} g^{(k)}_{1,1} & g^{(k)}_{1,2} & \cdots & g^{(k)}_{1,n}\\ g_{2,1^{(k)}} & g^{(k)}_{2,2} & & g^{(k)}_{2,n}\\ \vdots & & \ddots & \vdots\\ g^{(k)}_{m,1} & g^{(k)}_{m,2} & \cdots & g^{(k)}_{m,n}\end{pmatrix}$$

then $\mathrm{Trace}(g^k)=\sum_i \mathrm{Trace}(g^{(k)}_{i,i})$ and for each i, $g^{(k)}_{i,i}$ is a linear combination of $W(p)$'s with $tp=hp=i$. In particular $\mathrm{Trace}(g^{(k)}_{i,i}$ is a linear combination of $\mathrm{Trace}(W(p))$'s with p a loop. □

2. EXERCISES

Exercise 1. (a). Prove the well-known result in symmetric functions that $\lambda_1\dots\lambda_n$ is a polynomial in the power sums $p_k:=\lambda_1^k+\lambda_2^k+\cdots+\lambda_n^k$ with $k=1,2,\dots,n$. (Hint: In the formal power series ring $\mathbb{Q}[\lambda_1,\dots,\lambda_n][[t]]$, let $F(t)=(1-\lambda_1 t)(1-\lambda_2 t)\cdots(1-\lambda_n t)$ and let $G(t)=\log(F(t))$. Show that $G(t)=-\sum_{i=1}^{\infty}\frac{p_i t^i}{i}$ and use $F(t)=\exp(G(t))$.)

(b). Let A be a square $n\times n$ matrix with coefficients in a field K of characteristic 0. Show that $\det(A)$ is a polynomial in $\mathrm{Trace}(A^k)$ for $k=1,2,\dots,n$.