LECTURE 11: APPLICATION OF GEOMETRIC INVARIANT THEORY TO QUIVERS

HARM DERKSEN

1. Geometric Invariant Theory

Let us first discuss Geometric Invariant Theory (or GIT for short) and then we will apply this theory to representations of quivers. One well-known reference for GIT is


This book is not so suitable as an introduction, however. Other good sources are:


and


Suppose that $G$ is a reductive group acting on a projective variety $X$. The goal of Geometric Invariant Theory is to make "nice" quotient $\pi : X \to X/G$ for this action. Here "nice" may actually mean different things. The nicest type of quotient is a so-called geometric quotient. One of the main properties of a geometric quotient $\pi : X \to X/G$ is that the $G$-orbits in $X$ are exactly the fibers of $\pi$. Another notion of a quotient is a so-called good quotient which roughly means that for any $U \subset X/G$ open, the ring of regular functions on $U$ is exactly the ring of $G$-invariant regular functions on $\pi^{-1}(U)$. Geometric Invariant Theory gives a method for constructing such quotients. In general GIT does not construct a quotient for $X$ but rather a quotient of a suitable dense open subset of $X$. In fact GIT construct open subsets $X^s$ and $X^{ss}$ of $X$ such that $X^{ss}$ has a good quotient and this quotient is a projective variety, and $X^s$ has a geometric quotient. For simplicity, we will only consider varieties of the form $X = \mathbb{P}(V)$ where $V$ is a representation of $G$.

Definition 1. A 1-parameter subgroup (1-PSG) is a homomorphism $\lambda : K^* \to G$ of algebraic groups.
One-parameter subgroups play a role in the useful Hilbert-Mumford criterion below. We would like to explain what exactly we mean by the notation \( \lim_{t \to 0} \lambda(t) \cdot x \) with \( x \in V \) and \( \lambda : K^* \to G \) a 1-PSG. Of course if \( K = \mathbb{C} \), the complex numbers, this notation makes sense because \( \mathbb{C} \) has a topology. We will explain what the notation means for an arbitrary field \( K \).

If \( f \) is a Laurent-polynomial with coefficients in \( K \), i.e., \( f \in K[t, t^{-1}] \) then we define

\[
\lim_{t \to 0} f(t) = f(0)
\]

if \( f \in K[t] \) and

\[
\lim_{t \to 0} f(t) = \infty
\]

if \( f \not\in K[t] \). Choose a basis of \( V \) and suppose that \( x \in V \). Then the coordinates of \( \lambda(t) \cdot x \) are Laurent-polynomials in \( t \). We say \( \lim_{t \to 0} \lambda(t) \cdot x = \infty \) if at least one of the coordinates diverges, otherwise \( \lim_{t \to 0} \lambda(t) \cdot x \) is defined by the coordinatewise convergence.

**Theorem 1.** Suppose \( x \in V \) and \( V \) is a representation of a reductive group \( G \).

The following statements are equivalent:

(a). The orbit closure \( \overline{Gx} \) contains 0.

(b). For all \( f \in K[V]^G \) homogeneous of positive degree we have \( fxv) = 0 \).

(c). (Hilbert-Mumford criterion) There exists a 1-parameter subgroup \( \lambda : K^* \to G \) with \( \lim_{t \to 0} \lambda(t) \cdot x = 0 \).

**Definition 2.** An element \( x \in V \) is called **semi-stable** if the orbit closure \( \overline{Gx} \) does not contain 0. An element \( x \in V \) is called **stable** if the orbit \( \overline{Gx} \) is closed of dimension \( \dim(G) \) and \( x \neq 0 \).

Note that stability implies semi-stability. The set of semi-stable points \( x \in V \) is denoted by \( V^{ss} \) and the set of stable points \( x \in V \) is denoted by \( V^s \). Obviously \( V^{ss} \subseteq V^s \subseteq V \) and \( V^{ss}, V^s \) are open subsets of \( V \). If \( V \neq 0 \), then \( V^{ss} \) is also dense (but \( V^s \) does not need to be).

As before, let \( V//G \) be the affine variety corresponding to the invariant ring \( K[V]^G \). The inclusion \( K[V]^G \to K[V] \) defines a morphism \( \pi : V \to V//G \). Now we can define \( \mathbb{P}(V//G) \) as the projective variety corresponding to the graded ring \( K[V]^G \), i.e., \( \mathbb{P}(V//G) := \text{Proj}(K[V]^G) \). The morphism \( \pi : V \to V//G \) induces a rational map \( \mathbb{P}(V) \to \mathbb{P}(V//G) \) which is not a morphism in general, because the zero fiber \( \pi^{-1}(0) \) may contain nonzero elements of \( V \). Note that \( V^{ss} = V \setminus \{ \pi^{-1}(0) \} \) (follows from the definition and the Theorem). Now \( \pi \) induces a morphism \( \mathbb{P}(V^{ss}) \to \mathbb{P}(V//G) \) which is a “good quotient”. The restriction of \( \pi \) to \( \mathbb{P}(V^s) \) is a geometric quotient (onto the image in \( \mathbb{P}(V//G) \)).
2. Application of GIT to quiver representations

We now follow the paper


Let $Q$ be a quiver, $\alpha$ be a dimension vector and let $\sigma$ be a weight satisfying $\sigma(\alpha) = 0$. To a weight $\sigma : Q_0 \rightarrow \mathbb{Z}$ we will associate the character of $\text{GL}(\alpha)$ defined by

$$\{A(x) \mid x \in Q_0\} \in \text{GL}(\alpha) \mapsto \prod_{x \in Q_0} \det(A(x))^{\sigma(x)}.$$  

which, by abuse of notation, also will be denoted by $\sigma$. We define

$$\text{GL}(\alpha)_\sigma = \ker(\sigma).$$

Note that for any $t \in K^*$, $t \iden = \{t \iden_{\alpha(x)} \mid x \in Q_0\} \in \text{GL}(\alpha)_\sigma$ acts trivially on $\text{Rep}(Q, \alpha)$. This gives us a subgroup $K^* \subset \text{GL}(\alpha)_\sigma$ which acts trivially on $\text{Rep}(Q, \alpha)$, so we may view $\text{Rep}(Q, \alpha)$ as a representation of $G_\sigma = \text{GL}(\alpha)_\sigma/K^*$. Now we apply GIT to $G_\sigma$ acting on the representation $\text{Rep}(Q, \alpha)$ and acting on $\mathbb{P}(\text{Rep}(Q, \alpha))$. We define $V \in \text{Rep}(Q, \alpha)$ to be $(\sigma$-$(\text{semi})$)-stable if and only if $V \in \text{Rep}(Q, \alpha)$ is $(\text{semi})$-stable with respect to the action of $G_\sigma$. In other words:

**Definition 3.** (a) $V \in \text{Rep}(Q, \alpha)$ is called $\sigma$-semi-stable if and only if

$$0 \notin G_\sigma \cdot V = \overline{\text{GL}(\alpha)_\sigma \cdot V}.$$  

(b) $V \in \text{Rep}(Q, \alpha)$ is called $\sigma$-stable if $V \neq 0$ and $G_\sigma \cdot V = \text{GL}(\alpha)_\sigma \cdot V$ is a closed orbit of dimension

$$\dim G_\sigma = \dim \text{GL}(\alpha)_\sigma - 1 = \dim \text{GL}(\alpha) - 2.$$  

Note that

$$K[\text{Rep}(Q, \alpha)]^{G_\sigma} = K[\text{Rep}(Q, \alpha)]^{\text{GL}(\alpha)_\sigma} = \bigoplus_{m \geq 0} \text{SI}(Q, \alpha)_{m\sigma}.$$  

Like in GIT, we have the following theorem.

**Theorem 2.** The following statements are equivalent:

(a) $V$ is not $\sigma$-semi-stable, i.e., $\text{GL}(\alpha)_\sigma \cdot V$ contains 0.

(b) For all $m > 0$ and all $f \in \text{SI}(Q, \alpha)_{m\sigma}$ we have $f(V) = 0$.

(c) There exists a 1-PSG $\lambda : K^* \rightarrow \text{GL}(\alpha)_\sigma$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot V = 0$.

Suppose that $\lambda : K^* \rightarrow \text{GL}(\alpha)$ is a 1-PSG and $\sigma : \text{GL}(\alpha) \rightarrow K^*$ is a character. The composition $\sigma \circ \lambda : K^* \rightarrow K^*$ is an algebraic group homomorphism of $K^*$ to itself. We have $\sigma \circ \lambda(t) = t^k$ for some $k \in \mathbb{Z}$ and we define $(\sigma, \lambda) := k$.

King gave a more explicit description of semi-stable and stable representations $V$ in terms of subrepresentations of $V$:
Theorem 3 (A. King). Assume that $V$ is an $\alpha$-dimensional representation and \( \sigma(\alpha) = 0 \) for some weight $\sigma$. The following statements are equivalent:

(a). $V$ is $\sigma$-semistable;
(b). For every subrepresentation $V' \subseteq V$ we have $\sigma(d_{V'}) \leq 0$ where $d_{V'}$ is the dimension vector of $V'$.
(c). For every 1-PSG $\lambda : K^* \to \text{GL}(\alpha)$ for which $\lim_{t \to 0} \lambda(t) \cdot V$ exists we have $(\sigma, \lambda) \leq 0$.

Proof. (c)$\Rightarrow$(a) Suppose that $V$ is not $\sigma$-semi-stable. Then there exists a 1-PSG $\mu : K^* \to \text{GL}(\alpha)_\sigma$ such that $\lim_{t \to 0} \mu(t) \cdot V = 0$. We can find a 1-PSG $\lambda : K^* \to \text{GL}(\alpha)$ such that at each vertex $x$, $\lambda(t)(x)$ is a multiple of the identity and $(\sigma, \lambda) < 0$ (for example, find a vertex $x$ with $\sigma(x) \neq 0$, and define $\lambda(t)(x) = t^{-\sigma(x)} \text{id}$ and $\lambda(t)(y) = \text{id}$ for all $y \neq x$, then $(\sigma, \lambda) = -\sigma(x)^2 < 0$). For $k >> 0$ we have $\lambda t^k(t) \cdot V = 0$ (in particular it exists) and

$$(\sigma, \lambda t^k) = (\sigma, \lambda) + k(\sigma, \mu) = (\sigma, \lambda) < 0.$$ 

Contradiction with (c), so $V$ is $\sigma$-semi-stable.

(a)$\Rightarrow$(c) If $V$ is $\sigma$-semi-stable, then $f(V) \neq 0$ for some $f \in \text{SI}(Q, \alpha)m^\sigma$ with $m > 0$. Suppose that $\lim_{t \to 0} \lambda(t) \cdot V$ exists. Then

$$\lim_{t \to 0} f(\lambda(t) \cdot V) = f(\lim_{t \to 0} \lambda(t) \cdot V),$$ 

in particular, the limit exists. On the other hand,

$$\lim_{t \to 0} f(\lambda(t) \cdot V) = \lim_{t \to 0} (\lambda(t)^{-1} \cdot f)(V) = \lim_{t \to 0} t^{-m(\sigma, \lambda)} f(V).$$

and this shows that $(\sigma, \lambda) \leq 0$.

(c)$\Rightarrow$(b) Assume (c). Suppose $V' \subseteq V$ is a subrepresentation. For each vertex $x \in Q_0$ we can find a complement $V''(x)$ to $V'(x)$ in $V(x)$, i.e., we have $V(x) = V'(x) \oplus V''(x)$ for all $x \in Q_0$. Now we define $\lambda : K^* \to \text{GL}(\alpha)$ as follows. The element $\lambda(t)$ acts by multiplication by $t$ on $V'(x)$ for all $x \in Q_0$ and $\lambda(t)$ acts by multiplication by 1 on $V''(x)$ for all $x \in Q_0$. Let us see how $\lambda(t)$ acts on $V(a) : V'(ta) \oplus V''(ta) \to V'(ha) \oplus V''(ha)$. The map $V(a)$ is given by the block matrix

$$V(a) = \begin{pmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{pmatrix}. $$

Note that $A_{2,1} = 0$ because $V(a)(V'(ta)) \subseteq V'(ha)$. Also $A_{1,1} = V'(a)$ and $A_{2,2} = (V/V')(a)$. Now $\lambda(t)$ acts on the left and on the right on $V(a)$:

$$\lambda(t) \cdot V(a) = \begin{pmatrix} t \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} \begin{pmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{pmatrix} \begin{pmatrix} t^{-1} \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} = \begin{pmatrix} A_{1,1} & tA_{1,2} \\ 0 & A_{2,2} \end{pmatrix}. $$

This shows that $\lim_{t \to 0} \lambda(t) \cdot V(a)$ exists, so $\lim_{t \to 0} \lambda(t)V$ exists and the limit is $V' \oplus V/V'$.

(b)$\Rightarrow$(c) Suppose that $\lim_{t \to 0} \lambda(t) \cdot V$ exists. Via $\lambda$, also $K^*$ acts on the vector spaces $V(x)$, $x \in Q_0$. Let $V^{(k)}(x) \subseteq V(x)$ be the subspace of all $v \in V(x)$
with \( \lambda(t) \cdot v = t^n v \). For \( K^* \), every regular representation is a sum of irreducible one-dimensional representations. It follows that \( V(x) = \bigoplus_{k \in \mathbb{Z}} V(x)^{(k)} \). For every \( a \in Q_1 \) we can write \( V(a) \) as a block matrix \( (V(a)^{(k,l)})_{k,l} \) where \( V(a)^{(k,l)} : V(ta)^{(k)} \to V(\text{ha})^{(l)} \). Note that \( \lambda(t) \) acts on \( V(a)^{(k,l)} \) as multiplication by \( t^{k-l} \). Because \( \lim_{t \to 0} \lambda(t) \cdot V \) exists, we have \( V(a)^{(k,l)} = 0 \) if \( k > l \). Define \( V^{\geq m}(x) = \bigoplus_{k \geq m} V(x)^{(k)} \). It now follows that the spaces \( V^{\geq m}(x) \), \( x \in Q_0 \) actually define a subrepresentation \( V^{\geq m} \) because \( V(a) V^{\geq m}(ta) \subset V^{\geq m}(\text{ha}) \) for all \( a \in Q_1 \). Choose \( N \in \mathbb{Z} \) such that \( V(x)^{(k)} = 0 \) for \( k \leq N \) for all \( x \in Q_0 \). Now we have

\[
(\sigma, \lambda) = \sum_{x \in Q_0} \sum_{k=0}^{\infty} \sigma(x) k \dim V(x)^{(k)} = \sum_{x \in Q_0} \sum_{k=N+1}^{\infty} \sigma(x) \left( \left( \sum_{k=N+1}^{\infty} \dim V(x)^{(k)} \right) + N \dim V(x) \right) = \sum_{k=N+1}^{\infty} \sigma(d_V^{\geq k}) + N \sigma(\alpha) \leq 0
\]

because \( \sigma(\alpha) = 0 \) and \( \sigma(d_V^{\geq k}) \leq 0 \) for all \( k \). \( \square \)

**Theorem 4.** Let \( Q \) be a quiver, \( \alpha \) be a dimension vector and \( \sigma \) be a weight with \( \sigma(\alpha) = 0 \) and let \( V \) be a representation of dimension \( \alpha \). Then the following statements are equivalent:

(a) \( V \) is \( \sigma \)-stable.

(b) for every proper subrepresentation \( V' \subset V \) (i.e., \( d_{V'} \neq 0, \alpha \)) we have \( \sigma(d_{V'}) < 0 \).

**Lemma 1.** The map \( \text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta) \to \mathbb{Z} \) defined by

\[
(V, W) \to \dim \text{Hom}_Q(V, W)
\]

is semi-continuous.

**Proof.** We would like to show that the set of all pairs \( (V, W) \in \text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta) \) with \( \dim \text{Hom}_Q(V, W) \geq k \) is Zariski-closed. Consider the exact sequence

\[
0 \to \text{Hom}_Q(V, W) \to \bigoplus_{x \in Q_0} \text{Hom}(V(x), W(x)) \oplus \bigoplus_{a \in Q_1} \text{Hom}(V(ta), W(\text{ha}))
\]

Now \( \dim \text{Hom}_Q(V, W) \geq k \) if and only if \( d_W \) has rank \( q := \sum_{x \in Q_0} \alpha(x) \beta(x) - k \). This is equivalent with the vanishing of all \( q+1 \times q+1 \) minors of \( d_W \) which clearly defines a Zariski closed set. \( \square \)

**Proof of Theorem 4.** (a)\( \Rightarrow \) (b). Suppose that \( V \) is \( \sigma \)-stable. Suppose that \( V' \) is a proper subrepresentation of \( V \). For every vertex \( x \in Q_0 \) we can choose a complement \( V''(x) \) such that \( V(x) = V'(x) \oplus V''(x) \) for all \( x \in Q_0 \). Define a 1-parameter subgroup \( \lambda : K^* \to \text{GL}(\alpha)_\sigma \) such that \( \lambda(t) \) acts by multiplication by \( t \) on \( V'(x) \) and it acts trivially on \( V''(x) \). Then \( \lim_{t \to 0} \lambda(t) \cdot V = V' \oplus V/V' \) (as we
have seen before). We already know that $\sigma(d_{V'}) \leq 0$. If $\sigma(d_{V'}) = 0$, then $V'$ and $V/V'$ must be $\sigma$-semistable. Also $(\sigma, \lambda) = 0$ in that case so $\lambda$ is actually a 1-PSG in $GL(\alpha)$. Since the $GL(\alpha)_{\sigma}$-orbit of $V$ is closed, we get $V \cong V' \oplus V/V'$. There is a two-dimensional subgroup $(K^*)^2 \subseteq GL(\alpha)_{\sigma}$ where $(t, s)$ acts by multiplication with $t$ on $V'$ and by multiplication by $s$ on $V''$. This two-dimensional torus stabilizes $V \oplus V/V' \cong V$ and this shows that $\dim GL(\alpha)_{\sigma} \cdot V \leq \dim GL(\alpha)_{\sigma} - 2$. This shows that $V$ is not $\sigma$-stable, therefore, our assumption that $\sigma(d_{V'}) = 0$ was wrong, so $\sigma(d_{V'}) < 0$. (b)$\Rightarrow$(a). Assume (b). Suppose $V'$ lies in the Zariski closure of the $GL(\alpha)_{\sigma}$-orbit of $V$. Note that $V'$ is $\sigma$-semi-stable. By semi-continuity of Hom, we have $\text{Hom}_Q(V, V') \neq 0$ (since $\text{Hom}_Q(V, V) \neq 0$). Take $\phi : V \to V'$ non-trivial. The kernel $W$ of $\phi$ is also $\sigma$-semi-stable, in particular $\sigma(d_W) = 0$. This shows that $W = 0$ and $\phi$ is an isomorphism. We have proven that the $GL(\alpha)_{\sigma}$-orbit of $V$ is closed. A similar argument shows that if $\phi : V \to V$ is not an isomorphism, then $\phi = 0$. This shows that $\text{Hom}_Q(V, V) = K$. Now the $GL(\alpha)_{\sigma}$-orbit of $V$ has dimension $\dim(GL(\alpha)_{\sigma}) - 1$. We have proven that $V$ is $\sigma$-stable.

3. EXERCISES

**Exercise 1.** Show that if $\text{Rep}(Q, \alpha)$ has a dense $GL(\alpha)$-orbit, then $\text{Rep}(Q, m\alpha)$ has a dense $GL(m\alpha)$-orbit for any integer $m > 0$ (Let $V \in \text{Rep}(Q, \alpha)$ and recall that the codimension of the orbit of $V$ is exactly $\dim \text{Ext}_Q(V, V)$).

**Exercise 2.** Let $Q$ be a quiver and $\sigma$ be a weight. Suppose that $V$ and $W$ are $\sigma$-semi-stable representations and $\phi : V \to W$ is a morphism of quiver representations. Then the image, the kernel and the cokernel of $\phi$ are also $\sigma$-semi-stable. (You may not use Theorem 4, which wouldn’t be helpful anyway. This exercise is used in the proof of Theorem 4.)

**Exercise 3.** Let $Q$ be a quiver $\beta$ be a dimension vector. We say that a dimension vector $\beta$ is $\sigma$-(semi) stable if there exists an $\beta$-dimensional $\sigma$-(semi) stable representation (in that case a general representation of dimension $\beta$ is $\sigma$-(semi) stable).

We define

$$\Sigma(Q, \beta) = \{ \sigma \mid \beta \text{ is } \sigma\text{-semi-stable} \}.$$ 

(a). We write $\beta \leftrightarrow \beta$ if a general representation of dimension $\beta$ has a subrepresentation of dimension $\beta'$. Show that

$$\Sigma(Q, \beta) = \{ \sigma \mid \sigma(\beta) = 0 \text{ and } \sigma(\beta') \leq 0 \text{ for all } \beta' \leftrightarrow \beta \}.$$ 

(b). Show that $\beta$ is $\sigma$-stable if and only if $\sigma$ lies in the interior of $\Sigma(Q, \beta)$ (i.e., in the interior of the real cone $R_{+} \Sigma(Q, \beta)$ where $R_{+}$ are the positive real numbers).

(c). Prove that if $\dim SI(Q, \beta)_{\sigma} \leq 1$ for all weights $\sigma$, then $GL(\beta)$ has a dense orbit in $\text{Rep}(Q, \beta)$. The converse was already proven. (You may use the following facts from invariant theory: Suppose $G$ is a reductive group and
LECTURE 11: APPLICATION OF GEOMETRIC INVARIANT THEORY TO QUIVERS

$V$ is a representation of $V$. (1) If $G$ does not have a dense orbit in $V$ then the invariant field is nontrivial, i.e., if $K(V)^G \neq K$. (2) If $f, g \in K[V]$ are irreducible such that $f/g \in K(V)^G$, then $f$ and $g$ must be semi-invariants for $G$.)

(d). Show that if $\beta$ is $\sigma$-stable and dim $\text{SI}(Q, \beta)_{m\sigma} \leq 1$ for all $m$, then GL($\beta$) has a dense orbit in $\text{Rep}(Q, \beta)$.

(e). Suppose that $\sigma = \langle \alpha, \cdot \rangle$, and GL($\alpha$) has a dense orbit in Rep($Q, \alpha$) and $\beta$ is $\sigma$-stable, then GL($\beta$) has a dense orbit in Rep($Q, \beta$).

Exercise 4. Suppose that $Q$ is a quiver without oriented cycles and $\beta$ a dimension vector. Let $\mu = -\langle \cdot, \beta \rangle$. Show that $\text{SI}(Q, \beta)$ is generated by $c^\mu$’s such that $V$ is $\mu$-stable.

Exercise 5. Let $Q$ be a quiver without oriented cycles. We will now prove the following: If GL($\alpha$) has a dense orbit in Rep($Q, \alpha$), and $\sigma = \langle \alpha, \cdot \rangle$ then there are only finitely many $\sigma$-stable representations.

(a). Suppose that $W$ is a $\sigma$-stable representation of dimension $\beta$. Show that the orbit of $W$ is dense in $\text{Rep}(Q, \beta)$. If $W'$ is any other representation of dimension $\beta$ then either $W' \cong W$ or $W'$ is not $\sigma$-semi-stable. (Note that dim $\text{Hom}_Q(W, W) = 1$ has the minimal possible value over all representations in Rep($Q, \beta$), so $W$ has the largest possible orbit-dimension. Use the Exercise 3 to show that $W$ has a dense orbit. Then $W'$ lies in the orbit-closure of $W$. Show that $\text{Hom}_Q(W, W') \neq 0$.) In particular, there is only 1 $\beta$-dimensional $\sigma$-stable representation.

(b). Let $W$ be a $\sigma$-stable $\beta$-dimensional representation. Show that $c_W$ is irreducible (Suppose $c_W = c_{W_1}c_{W_2} = c_{W_1 \oplus W_2}$ then $W_1 \oplus W_2$ is $\sigma$-semi-stable).

(c). Let $W_1, \ldots, W_n$ be $\sigma$-stable representations. Show that $c_{W_1}, \ldots, c_{W_n}$ are algebraically independent (and in particular there are only finitely many $\sigma$-stable representations).