LECTURE 12: GENERAL REPRESENTATIONS OF QUIVERS

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In this lecture we will study "general representations", i.e., properties of representations which hold for some Zariski open subset of $\mathrm{Rep}(Q, \alpha)$. We will follow the paper
A. Schofield, *General representations of quivers*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 46–64.
Let us first introduce some useful notation.

1. NOTATION

Definition 1. We will write $\alpha \hookrightarrow \beta$ if a general representation of dimension β has an α-dimensional subrepresentation, i.e., there exists a Zariski open subset $U \subseteq \mathrm{Rep}(Q, \beta)$ such that every representation $V \in U$ has an α-dimensional subrepresentation. Similarly, we will write $\beta \twoheadrightarrow \alpha$ if a general representation of dimension β has an α-dimensional quotient representation.

We also define

$$\hom(\alpha, \beta) = \min\{\dim \mathrm{Hom}_Q(V, W) \mid V \in \mathrm{Rep}(Q, \alpha), W \in \mathrm{Rep}(Q, \beta)\}$$

and

$$\mathrm{ext}(\alpha, \beta) = \min\{\dim \mathrm{Ext}_Q(V, W) \mid V \in \mathrm{Rep}(Q, \alpha), W \in \mathrm{Rep}(Q, \beta)\}.$$

In the previous lecture we have seen that

$$(V, W) \mapsto \dim(\mathrm{Hom}_Q(V, W))$$

is a semicontinuous map. This shows that if $V \in \mathrm{Rep}(Q, \alpha)$ and $W \in \mathrm{Rep}(Q, \beta)$ are both in general position, then

$$\hom(\alpha, \beta) = \dim \mathrm{Hom}_Q(V, W), \qquad \mathrm{ext}(\alpha, \beta) = \dim \mathrm{Ext}_Q(V, W).$$

Also note that

$$\langle \alpha, \beta \rangle = \dim \mathrm{Hom}_Q(V, W) - \dim \mathrm{Ext}_Q(V, W) = \hom(\alpha, \beta) - \mathrm{ext}(\alpha, \beta).$$

We make the obvious observation

$$\alpha \hookrightarrow \beta \Leftrightarrow \beta \twoheadrightarrow \beta - \alpha,$$

because if a general representation of dimension β has an α-dimensional subrepresentation, then the quotient has dimension $\beta - \alpha$, and if a general β-dimensional representation has a $\beta - \alpha$-dimensional quotient, then the kernel of the quotient map is a subrepresentation of dimension α.

Let V be an n-dimensional vector space. Recall that $\widetilde{\mathrm{Gr}}(r,V)$ was the subvariety of $\bigwedge^r V$ of all

$$v_1 \wedge v_2 \wedge \cdots \wedge v_r$$

with $v_1,\dots,v_r \in V$. Then $\mathrm{Gr}(r,V) \subset \mathbb{P}(\bigwedge^r V)$ was the subvariety of $\mathbb{P}(\bigwedge^r V)$ corresponding to the cone $\widetilde{\mathrm{Gr}}(r,V)$. Points in $\mathrm{Gr}(r,V)$ correspond to r-dimensional subspaces of V. Choose a basis of $e_1,\dots,e_n$ of V. We define a linear function f on $\bigwedge^r V$ by $f(v_1 \wedge v_2 \wedge \cdots \wedge v_r)$ is the first minor of the $n \times r$ matrix

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_r \end{pmatrix}.$$

Let $U \subset \mathrm{Gr}(r,V)$ be the subset defined by $f \neq 0$ ($\mathrm{Gr}(r,V)$ is covered by similar open sets defined by other minors). If $W \in U \subset \mathrm{Gr}(r,V)$ (now thought of as an r-dimensional subspace) then we can choose a unique basis $v_1,\dots,v_r$ of W such that

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_r \end{pmatrix} = \begin{pmatrix} \mathrm{id}_r \\ A \end{pmatrix}$$

where A is some $n-r \times r$-matrix. So U can be identified with the set of $n-r \times r$ matrices. This shows that $\dim \mathrm{Gr}(r,V) = r(n-r)$. The tangent space of $\mathrm{Gr}(r,V)$ at W can be identified with $\mathrm{Hom}(W, V/W)$ (in fact, this can be done in a canonical way): Let $\varphi \in \hom(W,V/W)$, and let $i : W \to V$ be the inclusion. We can lift φ to $\widetilde{\varphi} : W \to V$ (but not in a unique way). Then for $t \in K$, $i + t\widetilde{\varphi} : W \to V$ defines an element in $\mathrm{Gr}(r,V)$ (as long as $i + t\widetilde{\varphi}$ is nondegenerate). By varying t we get a curve in $\mathrm{Gr}(r,V)$ and its derivative at $t=0$ gives a tangent vector at $W \in \mathrm{Gr}(r,V)$ which does not depend on the choice of $\widetilde{\varphi}$.

We define $\mathrm{Gr}(r,n) = \mathrm{Gr}(r,K^n)$. If Q is a quiver and α and β are dimension vectors with $\alpha(x) \leq \beta(x)$ for all $x \in Q_0$, then we define

$$\mathrm{Gr}(\alpha,\beta) = \prod_{x \in Q_0} \mathrm{Gr}(\alpha(x), \beta(x)).$$

So an element of $W \in \mathrm{Gr}(\alpha,\beta)$ can be viewed as a collection of subspaces $W(x) \subseteq K^{\beta(x)}$, $x \in Q_0$.

2. Some results of Schofield

Theorem 1. *The following statements are equivalent:*

(a). $\alpha \hookrightarrow \alpha + \beta$ *(i.e., a* **general** *representation of dimension $\alpha+\beta$ has an α-dimensional subrepresentation.*

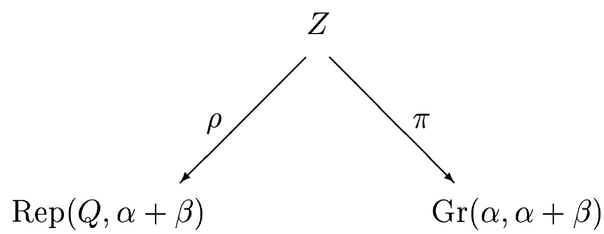
(b). **every** *representation of dimension $\alpha+\beta$ has an α-dimensional subrepresentation.*

(c). $\mathrm{ext}(\alpha,\beta) = 0$.

Proof. Let

$$Z \subset \mathrm{Rep}(Q, \alpha+\beta) \times \mathrm{Gr}(\alpha, \alpha+\beta)$$

be the set of all (V, W) with $V(a)W(ta) \subseteq W(ha)$ for all $a \in Q_1$. This means that the restriction of V to the subspaces $W(x)$, $x \in Q_0$ defines a subrepresentation of V, so instead of a collection of subspaces, W can be thought of as a subrepresentation of V. We consider the two projections



Suppose that $W \in \mathrm{Gr}(\alpha, \alpha+\beta)$ and consider the fiber $\pi^{-1}(W)$. Now $(V, W) \in \pi^{-1}(W)$ if and only if $V(a)W(ta) \subseteq W(ha)$. These are clearly linear conditions, and Z is a vector bundle over $\mathrm{Gr}(\alpha, \alpha+\beta)$. Let us compute the dimension of the fiber. For each arrow a we get $\dim W(ta)(\dim V(ha) - \dim W(ha)) = \alpha(ta)\beta(ha)$ linear conditions on the coefficients of the matrix $V(a)$, i.e., $V(a)$ lies in a space of dimension a $(\alpha+\beta)(ta)(\alpha+\beta)(ha) - \alpha(ta)\beta(ha)$. Therefore, the dimension of the fiber $\pi^{-1}(W)$ is equal to

$$\sum_{a \in Q_1} (\alpha+\beta)(ta)(\alpha+\beta)(ha) - \alpha(ta)\beta(ha).$$

The dimension of $\mathrm{Gr}(\alpha, \alpha+\beta) = \sum_{x \in Q_0} \alpha(x)\beta(x)$, so

$$\dim Z = \dim \mathrm{Gr}(\alpha, \alpha+\beta) + \dim \text{fiber} = \sum_{x \in Q_0} \alpha(x)\beta(x) +$$

$$+ \sum_{a \in Q_1} (\alpha+\beta)(ta)(\alpha+\beta)(ha) - \alpha(ta)\beta(ha) = \dim \mathrm{Rep}(Q, \alpha+\beta) + \langle \alpha, \beta \rangle.$$

Now we will consider the map ρ. Note that ρ is dominant if and only if $\alpha \hookrightarrow \alpha+\beta$. Let (V, W) be a general element of Z. Then we have $\dim \mathrm{Hom}_Q(V, W) = \hom(\alpha, \beta)$. We define $\mathrm{Gr}(\alpha, V) = \rho^{-1}(V)$ (this fiber is reduced because (V, W) is general). We can view $\mathrm{Gr}(\alpha, V)$ as a subset of $\mathrm{Gr}(\alpha, \beta)$ (but the do not have to be equal). Analogue to the result on ordinary Grassmannians, Schofield proved that the tangent space of $\mathrm{Gr}(\alpha, V)$ at (V, W) can be identified with $\mathrm{Hom}_Q(W, V/W)$. This shows that $\dim \mathrm{Gr}(\alpha, V) = \hom(\alpha, \beta)$. We have

$$\dim Z = \dim \operatorname{im} \rho + \dim \rho^{-1}(V)$$

So

$$\dim \mathrm{Rep}(Q, \alpha) + \langle \alpha, \beta \rangle = \dim \operatorname{im} \rho + \hom(\alpha, \beta)$$

and we get

$$\dim \mathrm{Rep}(Q, \alpha) - \dim \operatorname{im} \rho = \operatorname{ext}(\alpha, \beta).$$

This shows that ρ is dominant if and only if $\operatorname{ext}(\alpha, \beta) = 0$. On the other hand, from the definition of Z, ρ is dominant if and only if $\alpha \hookrightarrow \alpha+\beta$. $\square$

Lemma 1. *For a quiver Q and dimension vectors α, β, there exists an Zariski open dense subset $U \subset \mathrm{Rep}(Q,\alpha) \times \mathrm{Rep}(Q,\beta)$ and a dimension vector γ such that for $(V,W) \in U$ we have $\operatorname{rank} \varphi = \gamma$ for general $\varphi \in \mathrm{Hom}_Q(V,W)$. (The rank of φ is the dimension vector of the image of φ). We will call γ the general rank $\alpha \to \beta$.*

Proof. This is an exercise. $\square$

Theorem 2. *Let γ be the general rank $\alpha \to \beta$. Then*

$$\mathrm{ext}(\alpha,\beta) = -\langle \alpha - \gamma, \beta - \gamma \rangle = \mathrm{ext}(\alpha - \gamma, \beta - \gamma).$$

Proof. Note that $\gamma \hookrightarrow \beta$, so $\mathrm{ext}(\gamma, \beta - \gamma) = 0$. Suppose that $W \in \mathrm{Rep}(Q,\beta)$ is in general position. Now

$$\dim \mathrm{Gr}(\gamma, \beta - \gamma) = \hom(\gamma, \beta - \gamma) = \langle \gamma, \beta - \gamma \rangle.$$

We use the notation $\mathrm{Hom}(K^\alpha, W) = \prod_{x \in Q_0} \mathrm{Hom}(K^{\alpha(x)}, W(x))$. We now define

$$\mathrm{Hom}(K^\alpha, \gamma, W) \subseteq \mathrm{Hom}(K^\alpha, W) \times \mathrm{Gr}(\gamma, W)$$

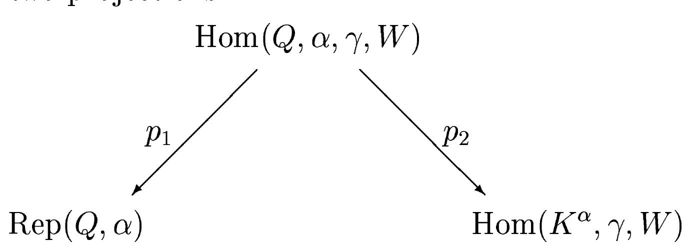
to be the set of all (φ, W') with $\operatorname{im} \varphi = W'$. Let $\psi : \mathrm{Hom}(K^\alpha, \gamma, W) \to \mathrm{Gr}(\gamma, W)$ be the projection. A fiber $\psi^{-1}(W')$ is the set of all surjective maps $\varphi \in \mathrm{Hom}(K^\alpha, W')$, so the dimension of the fiber $\psi^{-1}(W')$ is $\sum_{x \in Q_0} \alpha(x)\gamma(x)$. Therefore the dimension of $\mathrm{Hom}(K^\alpha, \gamma, W)$ is equal to

$$\langle \gamma, \beta - \gamma \rangle + \sum_{x \in Q_0} \alpha(x)\gamma(x).$$

Now we define

$$\mathrm{Hom}(Q,\alpha,\gamma,W) = \{(V, (\varphi, W') \mid \varphi : V \to W \text{ has image } W'\} \subset \mathrm{Rep}(Q,\alpha) \times \mathrm{Hom}(K^\alpha, \gamma, W)$$

We again have two projections:



Suppose $(\varphi, W') \in \mathrm{Hom}(K^\alpha, \gamma, W)$. Then the fiber $p_2^{-1}(\varphi, W')$ is the set of all V such that $\varphi(ha)V(a) = W(a)\varphi(ta)$. The dimension of the solution space of these linear equations in the coefficients of $V(a)$ is equal to the dimension of the solutions for $\varphi(ha)V(a) = 0$. Now $V(a) : V(ta) \to V(ha)$, so the dimension of the solution space is $\dim \ker(\varphi(ha)) \dim V(ta) = \alpha(ta)(\alpha - \gamma)(ha)$. This shows that the fibers of p_2 have dimension

$$\sum_{a \in Q_1} \alpha(ta)(\alpha - \gamma)(ha).$$

Finally the dimension of $\mathrm{Hom}(Q, K^\alpha, \gamma, W)$ is

(1)

$$\dim \mathrm{Hom}(K^\alpha, \gamma, W) + \dim \mathrm{fiber} = \langle \gamma, \beta - \gamma \rangle + \sum_{x \in Q_0} \alpha(x)\gamma(x) + \sum_{a \in Q_1} \alpha(ta)(\alpha - \gamma)(ha).$$

We can also compute the dimension of $\mathrm{Hom}(Q, K^\alpha, \gamma, W)$ using p_1. Let $V \in \mathrm{Rep}(Q, \alpha)$. Then

$$p_1^{-1}(V) \cong \{(\varphi, W') \mid \varphi : V \to W \text{ has image } W'\} \cong \{\varphi \mid \varphi : V \to W \text{ has rank } \gamma\}.$$

If V is general then the dimension of the fiber $p_1^{-1}(V)$ is equal to $\hom(\alpha, \beta)$. Therefore,

(2)

$$\dim \mathrm{Hom}(Q, \alpha, \gamma, W) = \dim \mathrm{Rep}(Q, \alpha) + \hom(\alpha, \beta) = \sum_{a \in Q_1} \alpha(ta)\alpha(ha) + \hom(\alpha, \beta).$$

By (1) and (2) we get:

$$\hom(\alpha, \beta) = \langle \gamma, \beta - \gamma \rangle + \sum_{x \in Q_0} \alpha(x)\gamma(x) + \sum_{a \in Q_1} \alpha(ta)(\gamma)(ha) = \langle \gamma, \beta - \gamma \rangle + \langle \alpha, \gamma \rangle.$$

So we get

$$\langle \alpha, \beta \rangle - \mathrm{ext}(\alpha, \beta) = \langle \gamma, \beta - \gamma \rangle + \langle \alpha, \gamma \rangle.$$

and finally

$$\mathrm{ext}(\alpha, \beta) = -\langle \alpha, \beta \rangle + \langle \gamma, \beta \rangle - \langle \gamma, \gamma \rangle + \langle \alpha, \gamma \rangle = -\langle \alpha - \gamma, \beta - \gamma \rangle.$$

We also get

$$\mathrm{ext}(\alpha, \beta) = -\langle \alpha - \gamma, \beta - \gamma \rangle \leq \mathrm{ext}(\alpha - \gamma, \beta - \gamma) \leq \mathrm{ext}(\alpha, \beta)$$

by the lemma below. □

Lemma 2. 1. *Suppose that* $\gamma \hookrightarrow \alpha$ *and* $\beta \twoheadrightarrow \delta$*, then*

$$\mathrm{ext}(\gamma, \beta) \leq \mathrm{ext}(\alpha, \beta).$$

and

$$\mathrm{ext}(\alpha, \delta) \leq \mathrm{ext}(\alpha, \beta).$$

In particular we have also

$$\mathrm{ext}(\gamma, \delta) \leq \mathrm{ext}(\gamma, \beta) \leq \mathrm{ext}(\alpha, \beta).$$

Proof. This follows from the long exact sequence and it is left as an exercise. □

Theorem 3. *We have*

$$\mathrm{ext}(\alpha, \beta) = \max_{\substack{\alpha' \hookrightarrow \alpha \\ \beta \twoheadrightarrow \beta'}} \{-\langle \alpha', \beta' \rangle\} = \max_{\alpha' \hookrightarrow \alpha} \{-\langle \alpha', \beta \rangle\} = \max_{\beta \twoheadrightarrow \beta'} \{-\langle \alpha, \beta' \rangle\}.$$

Proof. Since

$$-\langle \alpha', \beta' \rangle \leq \operatorname{ext}(\alpha', \beta') \leq \operatorname{ext}(\alpha, \beta)$$

we have

$$\max_{\substack{\alpha' \hookrightarrow \alpha \\ \beta \twoheadrightarrow \beta'}} \{-\langle \alpha', \beta' \rangle\} \leq \operatorname{ext}(\alpha, \beta).$$

On the other hand

$$\max_{\substack{\alpha' \hookrightarrow \alpha \\ \beta \twoheadrightarrow \beta'}} \{-\langle \alpha', \beta' \rangle\} \geq -\langle \alpha - \gamma, \beta - \gamma \rangle = \operatorname{ext}(\alpha, \beta)$$

where γ is the general rank $\alpha \to \beta$. We have shown that

$$\max_{\substack{\alpha' \hookrightarrow \alpha \\ \beta \twoheadrightarrow \beta'}} \{-\langle \alpha', \beta' \rangle\} = \operatorname{ext}(\alpha, \beta)$$

It is also clear that

$$\max_{\beta \twoheadrightarrow \beta'} \{-\langle \alpha, \beta' \rangle\} \leq \max_{\substack{\alpha' \hookrightarrow \alpha \\ \beta \twoheadrightarrow \beta'}} \{-\langle \alpha', \beta' \rangle\} = \operatorname{ext}(\alpha, \beta).$$

Note that

$$\operatorname{ext}(\alpha, \beta) \leq \operatorname{ext}(\alpha - \gamma, \beta - \gamma) \leq \operatorname{ext}(\alpha, \beta - \gamma) \leq \operatorname{ext}(\alpha, \beta)$$

by the lemma above, so $\operatorname{ext}(\alpha, \beta - \gamma) = \operatorname{ext}(\alpha, \beta)$. Let β' be a minimal (with respect to the partial ordering of dimension vectors) dimension vector such that

$$\operatorname{ext}(\alpha, \beta') = \operatorname{ext}(\alpha, \beta)$$

and $\beta \twoheadrightarrow \beta'$. Let δ be the general rank $\alpha \to \beta'$. Then similarly as before $\operatorname{ext}(\alpha, \beta') = \operatorname{ext}(\alpha, \beta' - \delta)$ and $\beta \twoheadrightarrow \beta' \twoheadrightarrow \beta' - \delta)$, so $\beta \twoheadrightarrow \beta' - \delta$. By minimality of β', we must have $\delta = 0$. This means that the general rank $\alpha \to \beta'$ is equal to 0, i.e., $\hom(\alpha, \beta') = 0$. But then $\operatorname{ext}(\alpha, \beta) = -\langle \alpha, \beta' \rangle$ and finally

$$\max_{\beta \twoheadrightarrow \beta'} \{-\langle \alpha, \beta' \rangle\} \geq \operatorname{ext}(\alpha, \beta)$$

so we have equality. In a similar fashion one can prove

$$\max_{\alpha' \hookrightarrow \alpha} \{-\langle \alpha', \beta \rangle\} = \operatorname{ext}(\alpha, \beta)$$

□