LECTURE 13: APPLICATION TO LITTLEWOOD-RICHARDSON COEFFICIENTS

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We will study the following set

$$\Sigma(Q,\beta)=\{\sigma \mid \mathrm{SI}(Q,\beta)_\sigma \neq 0\}.$$

We say that a dimension vector β is σ-(semi)-stable if and only if the general representation of dimension σ is indecomposable. From the definition follows that

$$\beta \text{ is } \sigma\text{-semi-stable} \Leftrightarrow \exists m>0\ \ \mathrm{SI}(Q,\beta)_{m\sigma}\neq 0 \Leftrightarrow \exists m>0\ m\sigma\in\Sigma(Q,\beta).$$

One question one might ask is whether the set $\Sigma(Q,\beta)$ is saturated: if $m\sigma\in\Sigma(Q,\beta)$ for some positive m, does it follow that $\sigma\in\Sigma(Q,\beta)$. We will show here that this is indeed the case. This will have an important implication on saturation of Littlewood-Richardson coefficients. From King's criterion of semi-stability we deduce

$$\exists m>0\, m\sigma\in\Sigma(Q,\beta)\Leftrightarrow \beta \text{ is } \sigma\text{-semi-stable} \Leftrightarrow \sigma(\beta)=0 \text{ and for all } \beta'\hookrightarrow\beta \text{ we have } \sigma(\beta')\leq 0.$$

We can prove a stronger result using Schofield's results in the previous lecture and the Theorem on spanning Schofield-semi-invariants. First we need the following Lemma:

Lemma 1. *Let Q be a quiver without oriented cycles. Suppose β is a dimension vector and σ is a weight. Assume we have $\sigma(\beta)=0$ and $\forall\beta'\hookrightarrow\beta$ we have $\sigma(\beta')\leq 0$. Then $\sigma=\langle\alpha,\cdot\rangle$ with α a dimension vector.*

Proof. We can write $\sigma\alpha,\cdot\rangle$ with $\alpha\in\mathbb{Z}^{Q_0}$ where α is unique. We want to prove that $\alpha\in\mathbb{N}^{Q_0}$. From the remarks above we see that β is σ-stable, so $\mathrm{SI}(Q,\beta)_{m\sigma}\neq 0$. By the Theorem on spanning Schofield semi-invariants there exists a nonzero $c^V\in\mathrm{SI}(Q,\beta)_{m\sigma}$. So we get $m\sigma=\langle\gamma,\cdot\rangle$, with $\gamma\in\mathbb{N}^{Q_0}$ is the dimension vector of V. This shows that $\alpha=\gamma/m$ and therefore it has nonnegative values on Q_0, so $\alpha\in\mathbb{N}^{Q_0}$. □

Theorem 1. *The set $\Sigma(Q,\beta)$ is saturated and given by*

$$\Sigma(Q,\beta)=\{\sigma\in\mathbb{Z}^{Q_0}\mid \sigma(\beta)=0 \text{ and } \forall\beta'\hookrightarrow\beta\ \sigma(\beta')\leq 0\}.$$

Proof. First note that

$$\sigma\in\Sigma(Q,\beta)\Leftrightarrow \mathrm{SI}(Q,\beta)_\sigma\neq 0\Leftrightarrow \exists V\ 0\neq c^V\in\mathrm{SI}(Q,\beta)_\sigma.$$

Now there exists such a nonzero c^V if and only if $\sigma = \langle \alpha, \cdot \rangle$ (α is the dimension of V) and $\hom(\alpha, \beta) = \operatorname{ext}(\alpha, \beta) = 0$. Now $\hom(\alpha, \beta) = \operatorname{ext}(\alpha, \beta) = 0$ is equivalent with $\langle \alpha, \beta \rangle = 0$ and $\operatorname{ext}(\alpha, \beta) = 0$. From the previous lecture, we know that

$$\operatorname{ext}(\alpha, \beta) = \max_{\beta \twoheadrightarrow \beta'} \{-\langle \alpha, \beta' \rangle\}.$$

This shows that $\operatorname{ext}(\alpha, \beta) = 0$ if and only if $\langle \alpha, \beta' \rangle \geq 0$ for all $\beta \twoheadrightarrow \beta'$. If we put $\beta = \beta' + \beta''$ and we assume that $\langle \alpha, \beta \rangle = 0$, then this is again equivalent with $\langle \alpha, \beta'' \rangle \leq 0$ for all $\beta'' \hookrightarrow \beta$. We conclude that

$$\sigma \in \Sigma(Q, \beta) \Leftrightarrow \sigma = \langle \alpha, \cdot \rangle \text{ with } \alpha \in \mathbb{N}^{Q_0},\ \sigma(\beta) = 0 \text{ and } \forall \beta'' \hookrightarrow \beta\ \sigma(\beta'') \leq 0.$$

Because of the Lemma, we may omit the condition that $\sigma = \langle \alpha, \cdot \rangle$ with α a dimension vector, since this follows automatically from the other conditions. □

Proposition 1. *Suppose that $\Sigma(Q, \beta)$ has maximal dimension $\#Q_0 - 1$. Then*

$$\Sigma(Q, \beta) = \{\sigma \mid \sigma(\beta) = 0 \text{ and } \hom(\beta', \beta - \beta') = \operatorname{ext}(\beta', \beta - \beta') = 0 \Rightarrow \sigma(\beta') \leq 0\}.$$

Proof. The condition on β is not necessary but we will use it in this proof. Note that $\operatorname{ext}(\beta', \beta - \beta') = 0$ if and only if $\beta' \hookrightarrow \beta$. Suppose that $\beta' \hookrightarrow \beta$ and $\hom(\beta', \beta - \beta') \neq 0$. Let γ be the generic rank $\beta' \to \beta - \beta'$. It follows that $\beta' - \gamma \hookrightarrow \beta$ and $\beta' + \gamma \hookrightarrow \beta$. Now

$$\sigma(\beta') = \frac{1}{2}(\sigma(\beta' + \gamma) + \sigma(\beta' - \gamma)) \leq 0$$

This shows that the inequality $\sigma(\beta') \leq 0$ on the set $\{\sigma \mid \sigma(\beta) = 0\}$ is redundant, or equivalent to $\sigma(\beta' - \gamma) \leq 0$. Note that $\beta' - \gamma$ is a smaller dimension vector. We keep repeating the process and we find that $\sigma(\beta') \leq 0$ is either redundant, or equivalent with an inequality $\sigma(\beta'') \leq 0$ with $\hom(\beta'', \beta - \beta'') = 0$. □

1. LR-coefficients and eigenvalues of Hermitian Matrices

Recall that to each partition λ with at most n parts we can associate an irreducible polynomial representation $S^\lambda(V)$ of the group $\operatorname{GL}(V)$. The set of all irreducible representations of $\operatorname{GL}(V)$ can be encoded by an n-tuple of integers $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We can define $c^\nu_{\lambda,\mu}$ for all $\lambda, \mu, \nu \in \mathbb{Z}^n$. We define $c^\nu_{\lambda,\mu} = 0$ if one of λ, μ, ν is not weakly decreasing. Let us consider the set K_n defined by

$$K_n = \{(\lambda, \mu, \nu) \in (\mathbb{Z}^n)^3 \mid c^\nu_{\lambda,\mu} \neq 0\}.$$

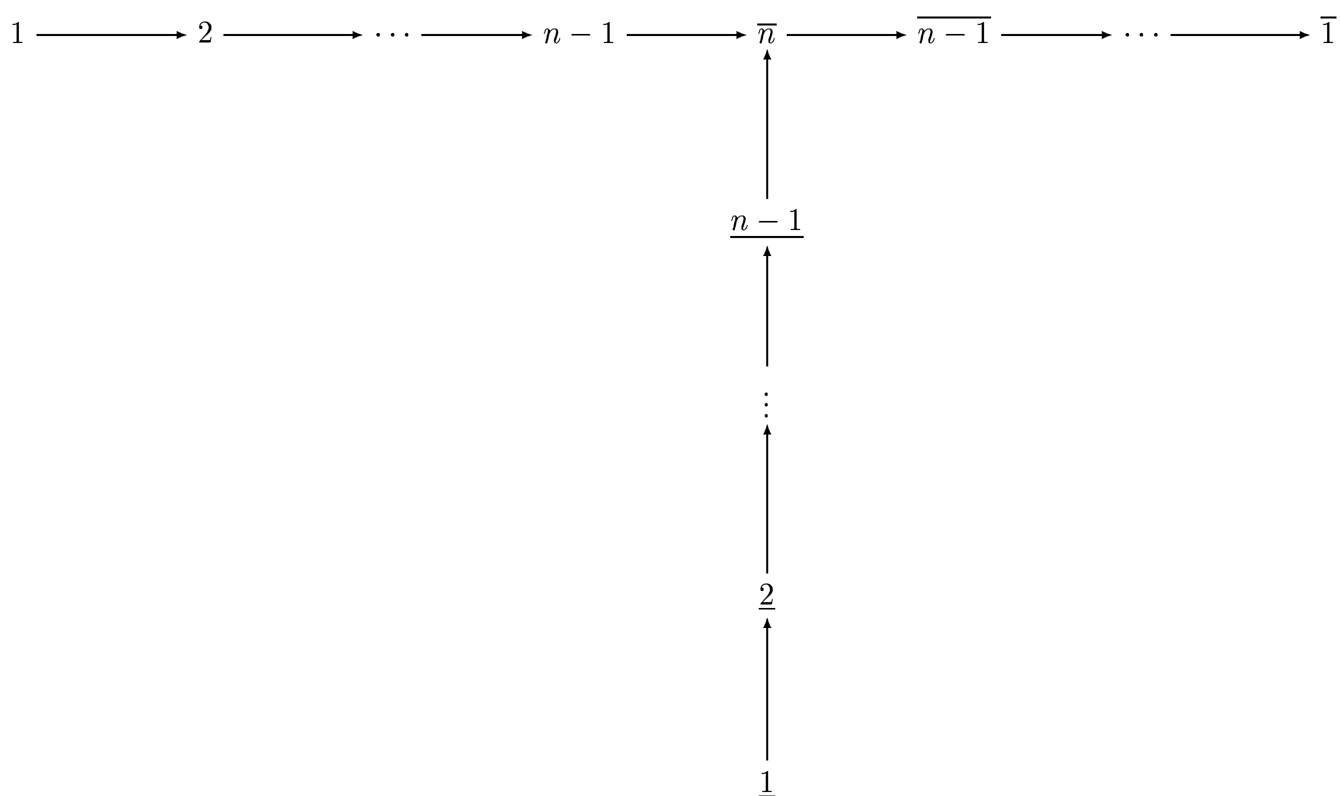
Let us for a moment discuss a seemingly unrelated problem. Let A, B, C be Hermitian matrices with $C = A + B$. Let $\lambda_1 \lambda_2 \cdots \geq \lambda_n$ be the eigenvalues of A, $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of B and $\nu_1 \geq \cdots \geq \nu_n$ be the eigenvalues of C. Let D_n be defined by

$$D_n = \{(\lambda, \mu, \nu) \in (\mathbb{R}^n)^3 \mid \text{there exist Hermitian matrices } A, B, C \text{ with prescribed eigenvalues}\}.$$

By results of Klyachko, the set D_n is just the $\mathbb{R}$-span of K_n and this set is given by a finite explicit set of inequalities. The set D_n has been studied for a long time. Various inequalities wer given. In particular Horn suggested an inductive procedure of constructing inequalities for D_n. To prove that these inequalities are necessary and sufficient, one needs to prove that the set K_n is saturated. This was recently proven by Knutson and Tao using the so-called combinatorial Honeycomb Model. For more details, see
W. Fulton, *Eigenvalues of sums of Hermitian matrices (after A. Klyachko)*, Séminaire Bourbaki. Vol. 1997/98. Astrisque No. **252** (1998), Exp. No. 845, 5, 255–269.
W. Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), no. 3, 209–249.
Using quiver representations we give here a totally different proof of the saturation of K_n.

2. Application to the triple flag quiver

Let us consider again the triple flag quiver $T^n_{n,n}$:



For β we take the dimension vector

$$\beta = \begin{array}{ccccccc} 1 & 2 & \cdots & n & \cdots & 2 & 1 \\ & & & \vdots & & & \\ & & & 2 & & & \\ & & & 1 & & & \end{array}$$

From the exercise in Lecture 8 we get that

$$\dim \mathrm{SI}(Q,\beta)_\sigma = c^{\nu}_{\lambda,\mu}$$

Where λ, μ, ν are given by

$$\begin{array}{rcl} \lambda' & = & ((n-1)^{\sigma(n-1)}, (n-2)^{\sigma(n-2)}, \cdots, 1^{\sigma(1)}) \\ \mu' & = & ((n-1)^{\sigma(\underline{n-1})}, (n-2)^{\sigma(\underline{n-2})}, \cdots, 1^{\sigma(\underline{1})}) \\ \nu' & = & (n^{-\sigma(\overline{n})}, (n-1)^{-\sigma(\overline{n-1})}, \cdots, 1^{-\sigma(\overline{1})}) \end{array}$$

where λ', μ', ν' are the dual partitions.

If $\lambda, \mu, \nu \in \mathbb{Z}^n$ then we can define the weight $\sigma^{\nu}_{\lambda,\mu}$ by

$$\begin{array}{rcll} \sigma^{\nu}_{\lambda,\mu}(i) & = & \lambda_i - \lambda_{i+1}, & \text{for } i = 1, 2, \ldots, n-1, \\ \sigma^{\nu}_{\lambda,\mu}(\underline{i}) & = & \mu_i - \mu_{i+1}, & \text{for } i = 1, 2, \ldots, n-1, \\ \sigma^{\nu}_{\lambda,\mu}(\overline{i}) & = & \nu_{i+1} - \nu_i, & \text{for } i = 1, 2, \ldots, n-1, \\ \sigma^{\nu}_{\lambda,\mu}(\overline{n}) & = & \lambda_n + \mu_n - \nu_n. & \end{array}$$

If $\tau = (1, 1, \ldots, 1)$ then

$$c^{\nu+(a+b)\tau}_{\lambda+a\tau,\mu+b\tau} = c^{\nu}_{\lambda,\mu}$$

for all $a, b \in \mathbb{Z}$ (note that $S^{\lambda+\tau}(V)$ and $S^{\lambda}(V)$ only differ by a character). Note also that

$$\sigma^{\nu+(a+b)\tau}_{\lambda+a\tau,\mu+b\tau} = \sigma^{\tau}_{\lambda,\mu}.$$

From all this, it is not hard to see that

$$(\lambda, \mu, \nu) \in K_n \Leftrightarrow \sigma_{\lambda,\mu,\nu} \in \Sigma(Q, \beta).$$

and

$$K_n \cong \mathbb{Z}^2 \times \Sigma(Q, \beta).$$

Corollary 1. *The set K_n is saturated because $\Sigma(Q, \beta)$ is saturated.*

We will now try to give a more precise description of the inequalities of the set K_n. Let us first do an example.

Example 1. Consider the quiver $T^3_{3,3}$ with dimension vector

$$\beta = \begin{array}{ccccc} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \\ & & 1 & & \end{array}.$$

Now it is not hard to see that

$$\beta' := \begin{matrix} 1 & 1 & 2 & 2 & 1 \\ & & 1 & & \\ & & 1 & & \end{matrix} \hookrightarrow \beta = \begin{matrix} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \\ & & 1 & & \end{matrix} .$$

The inequality $\sigma^{\nu}_{\lambda,\mu}(\beta') \leq 0$ gives

$$(\lambda_1-\lambda_2)+(\lambda_2-\lambda_3)+(\mu_1-\mu_2)+(\mu_2-mu_3)+(\nu_2-\nu_1)+2(\nu_3-\nu_2)+2(\lambda_3+\mu_3-\nu_3) \leq 0$$

so

$$\lambda_1+\lambda_3+\mu_1+\mu_3 \leq \nu_1+\nu_2.$$

Also note that

$$\beta'' := \begin{matrix} 0 & 0 & 0 & 1 & 0 \\ & & 0 & & \\ & & 0 & & \end{matrix} \hookrightarrow \beta = \begin{matrix} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \\ & & 1 & & \end{matrix} .$$

Now $\sigma^{\nu}_{\lambda,\mu}(\beta'') \leq 0$ gives

$$\nu_3-\nu_2 \leq 0.$$

Lemma 2. *The set $\Sigma(Q,\beta)$ has dimension $3n-2$.*

Proof. We will prove this in the future. The reason is that β is a so-called Schur root which will be explained in a future lecture. □

Proposition 2. *The set K_n is given by inequalities of the form*

$$\sum_{i\in I}\lambda_i+\sum_{j\in J}\mu_j \leq \sum_{k\in K}\nu_k \tag{1}$$

with $I,J,K \subseteq \{1,2,\ldots,n\}$ subsets of the same cardinality, and the obvious inequalities

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$
$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$$
$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$$

and the equality

$$\sum_{i}\lambda_i+\sum_{j}\mu_j = \sum_{k}\nu_k. \tag{2}$$

Proof. The set K_n is given by the inequalities $\sigma^{\nu}_{\lambda,\mu}(\beta') \leq 0$ with $\beta' \hookrightarrow \beta$. The condition $\sigma^{\nu}_{\lambda,\mu}(\beta)=0$ gives us (2). If β' is simple, it is easy to see that we must have $\beta'=\epsilon_{\bar{i}}$ for some i. We get the inequality $\nu_i \geq \nu_{i+1}$. If $\beta-\beta'$ is simple, it must be ϵ_i or $\epsilon_{\underline{i}}$ for some i and we get the inequality $\lambda_i \geq \lambda_{i+1}$ or $\mu_i \geq \mu_{i+1}$. Let us assume that β' and $\beta-\beta'$ are not simple. We claim that β' is weakly increasing along the arms (towards the center). If not, say if for example $\beta'(i+1) < \beta'(i)$, then any representation of dimension β' will be decomposable. In fact we can decompose $\beta'=\beta''+\beta'''$ with $\beta'' \hookrightarrow \beta'$ and $\beta''' \hookrightarrow \beta'$ (note: this could be explained better with the notion of canonical decomposition in a future lecture). Then $\sigma(\beta'') \leq 0$ and $\sigma(\beta''') \leq 0$ are also valid inequalities and the inequality

$$\sigma(\beta') = \sigma(\beta'') + \sigma(\beta''') \leq 0$$

is redundant. Similarly one can show that $\beta - \beta'$ must be weakly increasing. This shows that β' is weakly increasing along arms by steps at most 1. Now it is easy to see that $\sigma^{\nu}_{\lambda,\mu}(\beta') \leq 0$ has the desired form (1) with $|I| = |J| = |K| = \beta'(\overline{n})$. In fact we have

$$I = \{i \mid \beta'(i) > \beta'(i-)\}, J = \{j \mid \beta'(\underline{j}) > \beta'(\underline{j-1})\}, K = \{k \mid \beta'(\overline{k}) > \beta'(\overline{k-1})\}.$$

□

For a subset $I \subseteq \{1, 2, \dots, n\}$, say $I = \{i_1, i_2, \dots, i_r\}$ with $i_1 < i_2 < \cdots < i_r$ we define

$$\lambda(I) = (\lambda_r - r, i_{r-1} - r + 1, \cdots, i_1 - 1).$$

Proposition 3. *The set K_n is given by inequalities of the form (1) with*

$$c^{\lambda(K)}_{\lambda(I),\lambda(J)} \neq 0.$$

Proof. By Proposition 1 we only need to look at the inequalities

$$\sigma(\beta') \leq 0$$

with $\hom(\beta', \beta - \beta') = \operatorname{ext}(\beta', \beta - \beta') = 0$, i.e, $\beta' \circ \beta - \beta' > 1$. Put $\sigma = \langle \beta', \cdot \rangle$. Let us define $\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}$ by

$$\widetilde{\lambda}' = (\beta'(n-1)^{\sigma(n-1)}, \cdots, \beta'(1)^{\sigma(1)})$$
$$\widetilde{\mu}' = (\beta'(\underline{n-1})^{\sigma(\underline{n-1})}, \cdots, \beta'(\underline{1})^{\sigma(\underline{1})})$$
$$\widetilde{\nu}' = (\beta'(\overline{n})^{-\sigma(\overline{n})}, \cdots, \beta'(\overline{1})^{\sigma(\overline{1})})$$

Then $\sigma(i) = 1$ if $i+1 \in I$ and $\sigma(i) = 0$ otherwise. Also note that $\beta'(i_l - 1) = i_l - l$. This shows that $\widetilde{\lambda}' = \lambda(I)$. Similarly $\widetilde{\mu}' = \lambda(J)$. Note that $\sigma(\overline{k}) = -1$ if $k-1 \in K$ and $\sigma(\overline{k}) = 0$ otherwise. We also have $\beta'(\overline{k_l}) = k_l - l$ for all l. This shows that $\widetilde{\nu}' = \lambda(K)$. The condition $\beta' \circ (\beta - \beta') > 0$ is now equivalent with

$$c^{\widetilde{\nu}}_{\widetilde{\lambda},\widetilde{\mu}} = c^{\lambda(K)}_{\lambda(I),\lambda(J)} \neq 0.$$

□

Remark 1. It is known that we only need the inequalities with $c^{\nu}_{\lambda,\mu} = 1$. In fact this gives a set of inequalities which are necessary and sufficient. This was proven by Knutson, Tao and Woodward. However, a similar result can also be proven using quiver representations.