

LECTURE 2: THE CATEGORY OF QUIVER REPRESENTATIONS

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1. CATEGORIES AND THE KRULL-SCHMIDT THEOREM

Let us briefly discuss some basic notions of category theory. This language will be quite useful throughout the course.

Definition 1. A **category** \mathcal{C} is a *class* of objects $\text{Obj}(\mathcal{C})$, a set of **morphisms** $\text{Hom}_{\mathcal{C}}(A, B)$ for every ordered pair (A, B) of objects, an **identity morphism** $\text{id}_A \in \text{Hom}_{\mathcal{C}}(A, A)$ for every object, and a **composition map**

$$\text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$$

for every triple (A, B, C) of objects. We may also write $f : A \rightarrow B$ instead of $f \in \text{Hom}_{\mathcal{C}}(A, B)$. If $f : A \rightarrow B$ and $g : B \rightarrow C$ then the composition is denoted by gf or $g \circ f$. The above data is subject to the following axioms

1. $h(gf) = (hg)f$ if $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : A \rightarrow C$.
2. $\text{id}_B \circ f = f = f \circ \text{id}_A$ if $f : A \rightarrow B$.

Example 1. Sets form a category \mathcal{S} . The objects are sets. For every two sets A, B we define $\text{Hom}_{\mathcal{S}}(A, B)$ as the set of all maps $A \rightarrow B$. For every set A , $\text{id}_A : A \rightarrow A$ is the identity and the composition map $\text{Hom}_{\mathcal{S}}(A, B) \times \text{Hom}_{\mathcal{S}}(B, C) \rightarrow \text{Hom}_{\mathcal{S}}(A, C)$ is the usual composition of maps.

Example 2. Groups form a category. Objects are groups and morphisms are group homomorphisms. Vector spaces form a category where the objects are vector spaces and the morphisms are linear maps. For every ring R , we have a category of (left)- R -modules called $R\text{-mod}$.

Example 3. If Q is a quiver and K is a field, then we can define a category $\text{Rep}_K(Q)$ (or simply $\text{Rep}(Q)$) of representations of Q as follows. The objects are representations of Q and the morphisms are the morphisms as defined in the first lecture.

Definition 2. Suppose that A_1 and A_2 are objects of a category \mathcal{C} . An object B together with two maps $i_1 : A_1 \rightarrow B$ and $i_2 : A_2 \rightarrow B$ is called the **coproduct** or **direct sum** of A_1 and A_2 if it satisfies the following universal property: For every object C and every two morphisms $f_1 : A_1 \rightarrow C$, $f_2 : A_2 \rightarrow C$ there exists a unique morphism $f : B \rightarrow C$ such that $f_1 = f \circ i_1$ and $f_2 = f \circ i_2$.

Such a coproduct is unique up to isomorphism (Note: $\psi : A \rightarrow B$ is an isomorphism if there exists a morphism $\gamma : B \rightarrow A$ such that $\gamma \circ \psi = \text{id}_A$ and $\psi \circ \gamma = \text{id}_B$). We write $C = A_1 \oplus A_2$ (although this is only defined up to isomorphism). If in a category, every two objects have a coproduct, then we say that the category has (finite) coproducts.

Example 4. The direct sum for quiver representations as defined in the previous lecture is a coproduct in this categorical sense.

Definition 3. A category \mathcal{C} is called **additive** if

1. it has coproducts,
2. for every two objects A and B , $\text{Hom}_{\mathcal{C}}(A, B)$ has the structure of an abelian group and the composition map

$$\text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$$

is bilinear, and

3. the category has a zero object 0 . A zero object 0 is an object with the property that there is exactly one morphism from each object to 0 and exactly one morphism from 0 to any other object.

Note that in an additive category, we can define *indecomposable objects* in an obvious way.

Example 5. The category $\text{Rep}_K(Q)$ is additive. In fact, for every two representations V and W , $\text{Hom}_Q(V, W)$ is a K -vector space. For every three representations V , W and Z , the composition map

$$\text{Hom}_Q(V, W) \times \text{Hom}_Q(W, Z) \rightarrow \text{Hom}_Q(V, Z)$$

is K -bilinear. A category satisfying these additional properties is sometimes called a K -category.

We always will assume rings to have a 1. We will call a ring **local** if it has only one maximal right ideal. The following theorem shows that under mild conditions we have unique decomposition into indecomposables.

Theorem 1. *If \mathcal{C} is an additive category with finite coproducts in which every object is isomorphic to a finite direct sum of indecomposable objects, and for every object A the ring $\text{Hom}_{\mathcal{C}}(A, A)$ is local, then every object can be written **uniquely** (up to isomorphism/permutation of summands) as a direct sum of indecomposable objects.*

Proof. See: F. W. Anderson and K.R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics **13**, Springer-Verlag, Heidelberg-Berlin, 1973. \square

Corollary 1. *Let Q be a quiver and K be a field. In the category $\text{Rep}_K(Q)$ we have unique decomposition into indecomposables (up to permutation/isomorphism).*

Proof. We only have to show that for every representation V , then endomorphism ring $\text{Hom}_Q(V, V)$ is local. For every $\lambda \in K$ and $x \in Q_0$, define $V_\lambda(x)$ to be the generalized eigenspace of $\phi(x) : V(x) \rightarrow V(x)$ with eigenvalue λ , i.e.,

$$V_\lambda(x) = \{v \in V(x) \mid (\phi(x) - \lambda \text{id})^N v = 0 \text{ for } N \gg 0\}.$$

Note that $V(a)V_\lambda(ta) \subset V_\lambda(ha)$ because

$$(\phi(ha) - \lambda \text{id})^N V(a) = V(a)(\phi(ta) - \lambda \text{id})^N.$$

This shows that the $V_\lambda(x)$ actually define a subrepresentation V_λ of V . Since for every $x \in Q_0$, $V(x)$ is the direct sum of the generalized eigenspaces $V_\lambda(x)$, $\lambda \in K$ (we use that K is algebraically closed), we get that

$$V = \bigoplus_{\lambda \in K} V_\lambda$$

(only finitely many summands are nonzero). If V is indecomposable, then $V = V_\lambda$ for some λ . This shows that $\phi \in \text{Hom}_Q(V, V)$ is either nilpotent ($\lambda = 0$) or invertible ($\lambda \neq 0$).

Let \mathfrak{m} be the set of all nilpotent endomorphisms $\phi : V \rightarrow V$. Clearly, if $\psi \in \text{Hom}_Q(V, V)$ and $\phi \in \mathfrak{m}$ then $\psi\phi$ and $\phi\psi$ are both not invertible, so $\psi\phi, \phi\psi \in \mathfrak{m}$. Now \mathfrak{m} is also closed under addition because it is the kernel of the linear map $\text{Hom}_Q(V, V) \rightarrow K$ defined by

$$(1) \quad \phi \mapsto \frac{\sum_{x \in Q_0} \text{Trace}(\phi(x))}{\sum_{x \in Q_0} \dim V(x)}$$

(here we assume $\text{char}(K) \neq 0$). Now \mathfrak{m} is an ideal and (1) is a surjective ring homomorphism with kernel \mathfrak{m} . This shows that $\text{Hom}_Q(V, V)/\mathfrak{m} = K$ and therefore, \mathfrak{m} is a maximal ideal. \square

Neither the assumption $\text{char}(K) \neq 0$ nor the assumption that K is algebraically closed is really necessary (see Exercise 1). As we will see later, the category of representations of a quiver Q is equivalent to the category of representations of the so-called path algebra of Q (which we will define later). The unique decomposition theorem (Krull-Schmidt theorem) is often formulated for the category of modules over a ring, although more general versions exist.

2. FUNCTORS

Definition 4. A **covariant functor** $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}'$ maps any object A of \mathcal{C} to an object $\mathcal{F}(A)$ of \mathcal{C}' and for every two objects A and B we have a map

$$f \in \text{Hom}_{\mathcal{C}}(A, B) \mapsto \mathcal{F}(f) \in \text{Hom}_{\mathcal{C}'}(\mathcal{F}(A), \mathcal{F}(B))$$

such that $\mathcal{F}(\text{id}_A) = \text{id}_{\mathcal{F}(A)}$ for all objects A of \mathcal{C} and $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for all morphisms f and g in \mathcal{C} for which their composition makes sense. A **contravariant** functor is similar to a covariant functor, except that if $f : A \rightarrow B$

then $\mathcal{F}(f) : \mathcal{F}(B) \rightarrow \mathcal{F}(A)$ and $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for all $f : A \rightarrow B$ and $g : B \rightarrow C$.

Example 6. If \mathcal{C} is a category, then for every object A we have the functor \mathcal{F}_A from \mathcal{C} to the category of sets defined by

$$\mathcal{F}_A(B) = \text{Hom}_{\mathcal{C}}(A, B)$$

and for any morphism $f : B \rightarrow C$ we define

$$\mathcal{F}_A(f) : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$$

by

$$\mathcal{F}_A(f)g = fg.$$

This covariant functor \mathcal{F}_A is often denoted by $\text{Hom}_{\mathcal{C}}(A, \cdot)$.

Example 7. If \mathcal{C} is a category, then for every object A we have the functor \mathcal{F}^A from \mathcal{C} to the category of sets defined by

$$\mathcal{F}^A(B) = \text{Hom}_{\mathcal{C}}(B, A)$$

and for any morphism $f : B \rightarrow C$ we define

$$\mathcal{F}^A(f) : \text{Hom}_{\mathcal{C}}(C, A) \rightarrow \text{Hom}_{\mathcal{C}}(B, A)$$

by

$$\mathcal{F}^A(f)g = gf.$$

This contravariant functor \mathcal{F}^A is often denoted by $\text{Hom}_{\mathcal{C}}(\cdot, A)$.

If \mathcal{C} is an additive category, then we may view $\text{Hom}_{\mathcal{C}}(A, \cdot)$ and $\text{Hom}_{\mathcal{C}}(\cdot, A)$ as functors from \mathcal{C} to the category of abelian groups.

Definition 5. If \mathcal{C} is a category, then we can also define the opposite category \mathcal{C}^{op} . Objects of \mathcal{C}^{op} are the same as the objects of \mathcal{C} , but the arrows are reversed, i.e., we have a bijection

$$\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}^{\text{op}}}(B, A)$$

defined by

$$f \rightarrow f^{\text{op}}.$$

Composition in \mathcal{C}^{op} is defined by

$$g^{\text{op}} \circ f^{\text{op}} = (f \circ g)^{\text{op}}.$$

There is a natural contravariant functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ defined by $\mathcal{F}(A) = A$ for all objects A of \mathcal{C} and $\mathcal{F}(f) = f^{\text{op}}$ for all morphisms f in \mathcal{C} .

We still need to know when two categories are “isomorphic”. There is a small problem because two objects may be isomorphic but not identical. Let us first say what we mean by a natural isomorphism of functors.

Definition 6. A natural isomorphism between two functors $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}'$ and $\mathcal{G} : \mathcal{C} \rightarrow \mathcal{C}'$ is a rule that attaches to each object A of \mathcal{C} an isomorphism

$$\eta_A : \mathcal{F}(A) \rightarrow \mathcal{G}(A)$$

such that for every \mathcal{C} -morphism $f : A \rightarrow B$ the diagram

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) \\ \eta_A \downarrow & & \downarrow \eta_B \\ \mathcal{G}(A) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(B) \end{array}$$

i.e., $\eta_B \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_A$.

Now we are able to say when two categories are equivalent.

Definition 7. Two categories \mathcal{C} and \mathcal{C}' are equivalent if there are functors $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}'$ and $\mathcal{G} : \mathcal{C}' \rightarrow \mathcal{C}$ such that the composition functors $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ are equivalent to the identity functors on the categories \mathcal{C}' and \mathcal{C} respectively.

Example 8. The opposite quiver Q^{op} is the quiver (Q_0, Q_1^{op}) where

$$Q_1^{\text{op}} = \{a^{\text{op}} \mid a \in Q_1\}.$$

Now we define $ta^{\text{op}} = ha$ and $ha^{\text{op}} = ta$ (we reverse the directions of the arrows). The categories $\text{Rep}(Q^{\text{op}})$ and $\text{Rep}(Q)^{\text{op}}$ are equivalent.

3. KERNELS AND COKERNELS

Definition 8. Suppose that $Q = (Q_0, Q_1)$ is a quiver, K is a field and $\phi : V \rightarrow W$ is a morphism of quiver representations. The **kernel** $\ker(\phi)$ of ϕ is the subrepresentation of V defined by

$$\ker(\phi)(x) = \ker(\phi(x)) \subseteq V(x)$$

for all $x \in Q_0$. (Of course, $\ker(\phi)(a) : \ker(\phi(ta)) \rightarrow \ker(\phi(ha))$ is the restriction of $V(a)$ to $\ker(\phi(ta))$.)

Definition 9. The **image** $\text{im}(\phi)$ of ϕ is the subrepresentation of W defined by

$$\text{im}(\phi)(x) = \text{im}(\phi(x)) \subseteq W(x)$$

for all $x \in Q_0$. (Of course, $\text{im}(\phi)(a) : \text{im}(\phi(ta)) \rightarrow \text{im}(\phi(ha))$ is the restriction of $W(a)$ to $\text{im}(\phi(ta))$.)

Definition 10. If W is a subrepresentation of V , then we can define the quotient representation V/W by

$$(V/W)(x) = V(x)/W(x)$$

for all $x \in Q_0$. For every $a \in Q_0$, $V(a) : V(ta) \rightarrow V(ha)$ naturally induces a map

$$(V/W)(a) : V(ta)/W(ta) \rightarrow V(ha)/W(ha)$$

because $V(a)(W(ta)) \subseteq W(ha)$.

The **cokernel** of a map $\phi : V \rightarrow W$ is defined by

$$\text{coker}(\phi) = W/\text{im}(\phi).$$

This notion of kernel and cokernel for quiver representations are indeed kernels and cokernels in the categorical sense. More precisely: If $\phi : V \rightarrow W$ is a morphism of quiver representations, then $\ker(\phi) \hookrightarrow V$ (inclusion map) is a categorical kernel for $\phi : V \rightarrow W$ and $W \twoheadrightarrow \text{coker}(\phi)$ is a categorical cokernel for ϕ . In fact the category $\text{Rep}_K(Q)$ is an abelian category.

Definition 11. An sequence

$$V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} \cdots \xrightarrow{\phi_{n-1}} V_n$$

is called **exact** if $\text{im}(\phi_i) = \ker(\phi_{i+1})$ for $i = 1, \dots, n-2$.

Example 9. A sequence

$$0 \longrightarrow V \xrightarrow{\phi} W$$

is exact if and only if ϕ is injective (a monomorphism).

A sequence

$$V \xrightarrow{\phi} W \longrightarrow 0$$

is exact if and only if ϕ is surjective (an epimorphism).

Example 10. If

$$0 \longrightarrow W \longrightarrow V \longrightarrow X \longrightarrow 0$$

is exact, then $X \cong V/W$. Such a sequence is called a **short exact sequence**.

Definition 12. A (covariant) functor \mathcal{F} is exact, if for every exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

application of \mathcal{F} still gives an exact sequence

$$(2) \quad 0 \rightarrow \mathcal{F}(A) \rightarrow \mathcal{F}(B) \rightarrow \mathcal{F}(C) \rightarrow 0.$$

(A similar definition for contravariant functors with the arrows reversed in (2).)

Example 11. Suppose that Q is a quiver, K a field. If we apply the functor $\text{Hom}_Q(D, \cdot)$ to an exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

we obtain an exact sequence

$$0 \rightarrow \text{Hom}_Q(D, A) \rightarrow \text{Hom}_Q(D, B) \rightarrow \text{Hom}_Q(D, C).$$

However, the map $\text{Hom}_Q(D, B) \rightarrow \text{Hom}_Q(D, C)$ doesn't need to be surjective and the functor $\text{Hom}_Q(D, \cdot)$ might fail to be exact. If the functor $\text{Hom}_Q(D, \cdot)$ is exact, then D is called **projective**.

Example 12. Suppose that Q is a quiver, K a field. If we apply the functor $\text{Hom}_Q(\cdot, D)$ to the exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

then we obtain an exact sequence

$$0 \rightarrow \text{Hom}_Q(C, D) \rightarrow \text{Hom}_Q(B, D) \rightarrow \text{Hom}_Q(A, D).$$

Again, the map $\text{Hom}_Q(B, D) \rightarrow \text{Hom}_Q(A, D)$ doesn't need to be surjective and the contravariant functor $\text{Hom}_Q(\cdot, D)$ might fail to be exact. If the functor $\text{Hom}_Q(\cdot, D)$ is exact, then D is called **injective**.

Injective and projective objects play an important role in homological algebra as we will see later. In the next lecture, we will study the injective and projective quiver representations.

We end with some well-known easy facts about short exact sequences. Note that usually for an exact sequence

$$0 \rightarrow W \rightarrow V \rightarrow X \rightarrow 0$$

the representation V in the middle is not isomorphic to the direct sum of W and X . If there is a morphism $V \rightarrow W$ such that the composition $W \rightarrow V \rightarrow W = \text{id}_W$, then the exact sequence is said to be split. This is by the way equivalent with the existence of a morphism $X \rightarrow V$ such that the composition $X \rightarrow V \rightarrow X = \text{id}_X$ splits. If an exact sequence splits, then $V \cong W \oplus X$.

4. EXERCISES

Exercise 1. (a). Generalize the proof of Corollary 1 to the case where K has arbitrary characteristic. The problem is that (1) does not have \mathfrak{m} as its kernel. We have to show that \mathfrak{m} is closed under addition in a different way. Assume that $\phi^N = \psi^N = 0$. We would like to show that $\phi + \psi$ is also nilpotent. If not, then $\phi + \psi$ is invertible. Let $\alpha = \phi(\phi + \psi)^{-1}$ and $\beta = \psi(\phi + \psi)^{-1}$. Then $\alpha + \beta = \text{id}_V$ so in particular α and β commute. Show that $\text{id}^{2N} = (\alpha + \beta)^{2N} = 0$ (hence a contradiction). So \mathfrak{m} is closed under addition and is therefore an ideal. Show that \mathfrak{m} is a *maximal* ideal.

(b). Generalize the proof of Corollary 1 to the case where K is algebraically closed. For every irreducible polynomial $P(\lambda)$ one can define the space

$$V_P(x) = \{v \in V(x) \mid P(\phi(x))^N v = 0 \text{ for } N \gg 0\}.$$

Show that again that any $\phi \in \text{Hom}_Q(V, V)$ is invertible or nilpotent.

Exercise 2. Prove Theorem 1 for the category $\text{Rep}_K(Q)$:

(a). Suppose that

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r = W_1 \oplus W_2 \oplus \cdots \oplus W_s$$

where V is a representation of Q and $V_1, \dots, V_r, W_1, \dots, W_s$ are indecomposable representations. Let $i_k : V_k \hookrightarrow V$, $j_k : W_k \hookrightarrow V$ be the inclusions, and $p_k : V \rightarrow V_k$, $q_k : V \rightarrow W_k$ be the projections. Show that $p_1 j_k q_k i_1$ is an isomorphism for some k . (Hint: $\sum_k j_k q_k = \text{id}_V$ and use that $\text{Hom}_Q(V_1, V_1)$ is local).

(b). After a permutation of the W 's, we may assume that $k = 1$. So $p_1 j_1 q_1 i_1$ is an isomorphism. Show that $q_1 i_1$ is an isomorphism. (Hint: The exact sequence

$$0 \longrightarrow V_1 \xrightarrow{q_1 i_1} W_1 \longrightarrow W_1/V_1 \longrightarrow 0$$

splits.)

(c). Let $V'_2 = V_2 \oplus \cdots \oplus V_r$ and $W'_2 = W_2 \oplus \cdots \oplus W_s$. Let $i'_2 : V'_2 \hookrightarrow V$ and $q'_2 : V \rightarrow W'_2$ be the inclusion/projection. Show that $V'_2 \cong W'_2$. In fact this isomorphism $V_2 \rightarrow W_2$ is given by

$$q'_2 i'_1 - q'_2 i_1 (q_1 i_1)^{-1} q_1 i'_2.$$

Now the unique decomposition follows easily by induction.

Exercise 3. Let \mathcal{C} be an additive category.

- (a). Now $i : K \rightarrow A$ is called the kernel (in the categorical sense) of $f : A \rightarrow B$ if $f \circ i = 0$ and $f : A \rightarrow B$ is universal with this property, i.e., if $j : C \rightarrow A$ satisfies $f \circ j = 0$, then there exists a unique $g : C \rightarrow K$ such that $j = i \circ g$. Show that the kernel as defined for quiver representations is a kernel in this categorical sense.
- (b). Dually, we say that $p : B \rightarrow C$ is the cokernel of $f : A \rightarrow B$ if $p \circ f = 0$ and $p : B \rightarrow C$ is universal with this property, i.e., if $q : B \rightarrow D$ satisfies $q \circ f = 0$, then there exists a unique $g : C \rightarrow D$ such that $q = g \circ p$. Show that the cokernel as defined for quiver representations is a cokernel in this categorical sense.
- (c). A map $f : A \rightarrow B$ is called a monomorphism if for every $i_1 : K \rightarrow A$ and $i_2 : K \rightarrow A$ we have that $f \circ i_1 = f \circ i_2$ implies $i_1 = i_2$. Show that in the category $\text{Rep}_K(Q)$, monomorphisms are morphisms $\phi : V \rightarrow W$ where $\phi(x)$ is injective for all $x \in Q_0$.
- (d). A map $f : A \rightarrow B$ is called an epimorphism if for every $p_1 : B \rightarrow C$ and $p_2 : B \rightarrow C$ we have that $p_1 \circ f = p_2 \circ f$ implies $p_1 = p_2$. Show that in the category $\text{Rep}_K(Q)$, epimorphisms are morphisms $\phi : V \rightarrow W$ where $\phi(x)$ is surjective for all $x \in Q_0$.
- (e). An additive category \mathcal{C} is called abelian if:
- Every morphism has a kernel and cokernel,
 - every monomorphism is the kernel of its cokernel, and

(c) every epimorphism is the cokernel of its kernel.
Show that $\text{Rep}_K(Q)$ is an abelian category.

Exercise 4. Let Q be a quiver. Using dual spaces, define a functor $*$: $\text{Rep}(Q) \rightarrow \text{Rep}(Q^{\text{op}})$ ($V \mapsto V^*$) with the property that the composition with itself $**$: $\text{Rep}(Q) \rightarrow \text{Rep}(Q)$ is equivalent to the identity functor.