

# LECTURE 6: (SEMI)-INVARIANTS OF QUIVER REPRESENTATIONS

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## 1. INTRODUCTION

In this lecture we will discuss (polynomial or rational) invariants and semi-invariants of quiver representations. If for example the quiver is not of finite representation type (and there are infinitely many distinct representations for some dimensions), then we still may be able to distinguish representations using polynomial invariants. For large (so-called wild) quivers, a classification of indecomposable representations is practically impossible. However, one can still look at the structure of invariants. Using invariant rings one can construct moduli spaces which are algebraic varieties who (roughly speaking) parametrize the set of isomorphism classes of a given dimension vector. Let us first give some examples of invariants.

**Example 1.** Consider the quiver  $Q$  with one vertex (1) and one arrow ( $a$ ). Let  $\alpha = n$  be a dimension vector. Then  $\text{Rep}(Q, \alpha) = \text{Hom}(K^n, K^n)$ . One invariant polynomial function is

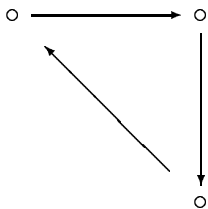
$$A \in \text{Rep}(Q, \alpha) \mapsto \det(A).$$

This is an invariant because  $\det(A)$  only depends on the isomorphism class of  $A \in \text{Rep}(Q, \alpha)$ . If we choose a different basis for  $K^n$ , then this would not change the value of  $\det(A)$ . In other words,  $\det(A)$  is invariant under the action of  $\text{GL}(\alpha) = \text{GL}(n)$  (which acts by conjugation on  $\text{Rep}(Q, \alpha)$ ). Besides  $\det(A)$  there are other invariants as well, for example the trace of a matrix. In fact, all coefficients of the characteristic polynomial

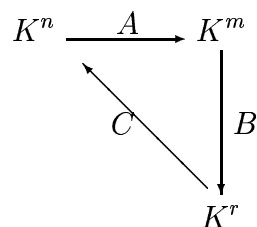
$$\det(\lambda \text{id} - A) = \lambda^n - s_1(A)\lambda^{n-1} + s_2(A)\lambda^{n-2} + \cdots + (-1)^n s_n(A)$$

are invariants (note that  $s_n(A) = \det(A)$  and  $s_1(A)$  is the trace of  $A$ ). In some sense these are all invariants. It is known that any polynomial invariant is a polynomial in  $s_1, s_2, \dots, s_n$ .

**Example 2.** Consider the quiver

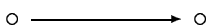


and the representation



then  $\det(CBA)$  is an invariant (which may be equal to 0), also  $\det(BAC)$  and  $\det(CAB)$  are invariants.

**Example 3.** Consider the quiver:



Let  $\alpha = (n, n)$ . Then  $\text{Rep}(Q, \alpha) = \text{Hom}(K^n, K^n)$  and  $\text{GL}(\alpha) = \text{GL}(n) \times \text{GL}(n)$  acts on  $\text{Hom}(K^n, K^n)$ . Now  $\det(A)$  is not an invariant, but a semi-invariant: A basis change will change  $\det(A)$  with the determinant of that base change.

## 2. INVARIANT THEORY

We will just introduce the basics of invariant theory. Suppose that  $X$  is an affine variety (i.e., a Zariski closed subset of  $K^n$  for some  $n$ ). The coordinate ring of  $X$  is denoted by  $K[X]$ . Let  $G$  be a linear algebraic group (i.e., a Zariski closed subgroup of  $\text{GL}(k)$  for some  $k$ ). We say that  $G$  acts **regularly** on  $X$  if there is a morphism of affine varieties

$$\rho : G \times X \rightarrow X$$

such that

$$g \cdot x := \rho(g, x), \quad g \in G, x \in X$$

satisfies the action axioms (i.e.,  $gh \cdot x = g \cdot (h \cdot x)$  for all  $g, h \in G$ ,  $e \cdot x = x$ ).

If  $G$  is a linear algebraic group action on  $X$ , then it also acts on the coordinate ring  $K[X]$  as follows:

$$g \cdot f(x) := f(g^{-1} \cdot x), \quad g \in G, f \in K[X], x \in X.$$

Note that we have to put the inverse on the right-handside, because otherwise it wouldn't be a (left) action.

An element  $f \in K[X]$  is called an **invariant** if

$$g \cdot f = f$$

for all  $g \in G$ . Let us denote the set of invariant polynomials by  $K[X]^G$ . It is easy to see that  $K[X]^G$  is a sub-algebra of  $K[X]$ . It is a classical problem in invariant theory whether such invariant rings  $K[X]^G$  can be generated by finitely many polynomials (see Hilbert's fourteenth problem). Because of results of Hilbert, Nagata and others, it is now known that  $K[X]^G$  is finitely generated if  $G$  is **reductive**. In characteristic 0, a group is reductive if and only if every representation is the direct sum of irreducible representations. Examples of reductive groups are  $GL(n)$ ,  $SL(n)$  and all other classical groups. The multiplicative group  $K^*$  is reductive, but the additive group  $K$  is not. If  $G$  is reductive (and therefore  $K[X]$  finitely generated), then we can view  $K[X]^G$  as the coordinate ring of an affine variety which we will denote by  $X//G$ . The inclusion  $K[X]^G \subset K[X]$  corresponds to a dominant morphism (in fact surjective) of affine varieties

$$\pi : X \rightarrow X//G.$$

This map  $\pi$  we can view as a quotient, although it may not always be true that the fibers of  $\pi$  are exactly the orbits.

Besides invariants, let us also discuss semi-invariants now. A **character** of an algebraic group  $G$  is a homomorphism

$$\chi : G \rightarrow K^*$$

of algebraic groups, i.e.,  $\chi$  is a morphism of affine varieties and a group homomorphism. The set of characters is an abelian group (by multiplication).

An element  $f \in K[X]$  is called a semi-invariant with character  $\chi$  if

$$g \cdot f = \chi(g)f, \quad g \in G.$$

Let us denote the set of semi-invariants with character  $\chi$  by  $K[X]_\chi$ . Let  $[G, G]$  be the **commutator subgroup** of  $G$  generated by all  $ghg^{-1}h^{-1}$  with  $g, h \in G$ . The group  $[G, G]$  is again an algebraic group, and so is the abelian group  $G/[G, G]$ .

**Lemma 1.** *Suppose that  $G$  is reductive and  $K$  has characteristic 0. We have equality*

$$(1) \quad K[X]^{[G, G]} = \bigoplus_{\chi} K[X]_\chi.$$

*Proof.* This is a well-known, but we give a **sketch** of the proof here. Clearly every character  $\chi$  is trivial on  $[G, G]$ . This proves  $\supseteq$ . Since  $G$  is reductive, so is  $G/[G, G]$  (since every representation of  $G/[G, G]$  is also a representation of  $G$  and is therefore a direct sum of irreducibles). Now  $G/[G, G]$  is abelian and reductive. From Schur's lemma follows that any irreducible representation of  $G/[G, G]$  must be one-dimensional (since every element commutes with the action, every element must act as  $\lambda \text{id}$  for some  $\lambda$ ). Let  $f \in K[X]^{[G, G]}$ . Then  $f$  is contained in some finite dimensional  $G$ -stable subspace  $W \subset K[X]^{[G, G]}$  and  $W$  is a representation of  $G$  (this is a consequence of the regularity of the action of  $G$  on  $X$ ). We can

write  $W = W_1 \oplus \cdots \oplus W_r$  with  $W_i$  irreducible and 1-dimensional. We can write  $f = f_1 + f_2 + \cdots + f_r$  with  $f_i \in W_i$ . Now every  $f_i$  is a semi-invariant. Also note the the righthandside of (1) is a direct sum, i.e., if  $f_1, \dots, f_r$  are nonzero semiinvariants with distinct characters, then  $f_1, \dots, f_r$  are linearly independent (see Exercise 2).  $\square$

### 3. INVARIANTS AND SEMIINVARIANTS FOR QUIVERS

We will now apply invariant theory to representations of quivers. We have the algebraic group  $\mathrm{GL}(\alpha)$  acting on the representation space  $\mathrm{Rep}(Q, \alpha)$ . We define the **ring of invariants** as

$$I(Q, \alpha) = K[\mathrm{Rep}(Q, \alpha)]^{\mathrm{GL}(\alpha)}.$$

The ring of invariants is trivial if  $Q$  has no oriented cycles (see Exercies 1). We will therefore also look at the ring of semi-invariants.

Suppose that  $\chi : \mathrm{GL}(\alpha) \rightarrow K^*$  is a character. Such a character always looks like

$$\{\phi(x) \in \mathrm{GL}(\alpha(x)) \mid x \in Q_0\} \mapsto \prod_{x \in Q_0} \det(\phi(x))^{\sigma(x)}.$$

Here  $\sigma : Q_0 \rightarrow \mathbb{Z}$  is called the weight. We will view weights as being dual to dimension vectors as follows. If  $\alpha$  is a dimension vector, then we define

$$\sigma(\alpha) = \sum_{x \in Q_0} \sigma(x)\alpha(x).$$

The **ring of semiinvariants** is defined as

$$\mathrm{SI}(Q, \alpha) = K[\mathrm{Rep}(Q, \alpha)]^{\mathrm{SL}(\alpha)}.$$

**Example 4.** Consider the quiver  $Q$ :

$$1 \xrightarrow{a} 2$$

and the dimension vector  $\alpha = (n, n)$  (i.e.,  $\alpha(1) = \alpha(2) = n$ ). Then  $\mathrm{Rep}(Q, \alpha) = \mathrm{Hom}(K^n, K^n)$  and  $\mathrm{GL}(\alpha) = \mathrm{GL}(n) \times \mathrm{GL}(n)$ . The action of  $\mathrm{GL}(n) \times \mathrm{GL}(n)$  on  $\mathrm{Rep}(Q, \alpha)$  is given by

$$(B_1, B_2) \cdot A = B_2 A B_1^{-1}.$$

If  $f \in K[\mathrm{Rep}(Q, \alpha)]$  then the action of  $\mathrm{GL}(n) \times \mathrm{GL}(n)$  on  $f$  is given by

$$(B_1, B_2) \cdot f(A) = f(B_2^{-1} A B_1)$$

(remember to put inverse here). If we take  $f = \det(A)$ , then we get

$$(B_1, B_2) \cdot \det(A) = \det(B_2^{-1} A B_1) = \det(B_1) \det(B_2)^{-1} \det(A).$$

So the character corresponding to the semi-invariant  $\det(A)$  is  $(B_1, B_2) \mapsto \det(B_1) \det(B_2)^{-1}$  and the weight is  $\sigma = (1, -1)$  (i.e.,  $\sigma(1) = 1$  and  $\sigma(2) = -1$ ).

**Remark 1.** If  $\mathrm{SI}(Q, \alpha)_\sigma \neq 0$ , then  $\sigma(\alpha) = 0$ .

*Proof.* Consider the action of  $\lambda \text{id} = \{\lambda \text{id}_{\alpha(x)} \mid x \in Q_0\} \in \text{GL}(\alpha)$ . First of all  $\lambda \text{id}$  acts trivially on  $\text{Rep}(Q, \alpha)$  and on  $K[\text{Rep}(Q, \alpha)]$ . If  $f$  is a semiinvariant of weight  $\sigma$ , then

$$f = (\lambda \text{id}) \cdot f = \prod_{x \in Q_0} \det(\lambda \text{id}_{\alpha(x)})^{\sigma(x)} f = \prod_{x \in Q_0} \lambda^{\sum_{x \in Q_0} \alpha(x) \sigma(x)} f.$$

This shows that  $\sigma(\alpha) = \sum_{x \in Q_0} \alpha(x) \sigma(x) = 0$ .  $\square$

**Proposition 1.** *Suppose that  $\text{GL}(\alpha)$  has a dense orbit in  $\text{Rep}(Q, \alpha)$ . Let  $S$  be the set of all  $\sigma$  such that there exists an  $f_\sigma \in \text{SI}(Q, \alpha)_\sigma$  which is nonzero and irreducible.*

- (a). *For every weight  $\sigma$  we have  $\dim \text{SI}(Q, \alpha)_\sigma \leq 1$ .*
- (b). *All weights in  $S$  are linearly independent over  $\mathbb{Q}$ .*
- (c).  *$\text{SI}(Q, \alpha)$  is the polynomial ring generated by all  $f_\sigma$ ,  $\sigma \in S$ .*

*Proof.* (a) Suppose that  $f, h \in \text{SI}(Q, \alpha)_\sigma$ . Since  $f/h$  is constant on the dense orbit, we must have  $f/h$  is constant and  $f$  and  $h$  must be linearly dependent. (b) Suppose that

$$\sum_{\sigma \in S} a_\sigma \sigma = 0$$

with  $a_\sigma \in \mathbb{Z}$  for all  $\sigma$ . Then we have

$$\sum_{a_\sigma > 0} a_\sigma \sigma = \sum_{a_\sigma < 0} |a_\sigma| \sigma.$$

and therefore

$$\prod_{a_\sigma > 0} f_\sigma^{a_\sigma} = \lambda \prod_{a_\sigma < 0} f_\sigma^{|a_\sigma|}$$

for some nonzero  $\lambda \in K$ . From unique factorization in  $K[\text{Rep}(Q, \alpha)]$  follows that  $a_\sigma = 0$  for all  $\sigma$ . (c) Every semi-invariant is a product of irreducible semiinvariants (see Exercise 3). This shows that  $f_\sigma$ ,  $\sigma \in S$  generate  $\text{SI}(Q, \alpha)$ . Also, all monomials in the  $f_\sigma$ 's have distinct weights, so all monomials in the  $f_\sigma$ 's are linearly independent (see Exercise 2). This shows that the  $f_\sigma$ 's are algebraically independent.  $\square$

#### 4. EXERCISES

**Exercise 1.** Show that if  $Q$  is a quiver without oriented cycles, then  $\text{I}(Q, \alpha) = K$ , i.e., there are no nontrivial invariants. Hint: Without loss of generality we may assume that  $Q_0 = \{1, 2, \dots, n\}$  and  $ta < ha$  for all arrows  $a$ . Define  $\phi_\lambda \in \text{GL}(\alpha)$  by

$$\phi_\lambda(k) = \lambda^k \text{id}_{\alpha(k)} \in \text{GL}(\alpha(x))$$

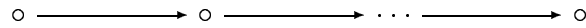
for  $k = 1, \dots, n$ . Consider the action of  $\phi_\lambda$  on  $\text{Rep}(Q, \alpha)$  and on  $K[\text{Rep}(Q, \alpha)]$ .

**Exercise 2.** Suppose that  $G$  is a linear algebraic group acting regularly on an affine variety. Suppose that  $f_1, f_2, \dots, f_r \in K[X]$  are nonzero semi-invariants with distinct characters  $\chi_1, \chi_2, \dots, \chi_r$ . Show that  $f_1, \dots, f_r$  are linearly independent.

**Exercise 3.** Suppose that a connected algebraic group  $G$  acts on an irreducible reduced affine variety  $X$ . If  $f$  is a semi-invariant, and  $h$  divides  $f$ , then  $h$  is also a semi-invariant. Hint: the ideal  $(f)$  is stable under the action of  $G$ . Use the uniqueness of decomposition into prime ideals to show that  $(h)$  is also stable under  $G$ .

**Exercise 4.** Suppose that  $\mathrm{GL}(\alpha)$  has a dense orbit in  $\mathrm{Rep}(Q, \alpha)$  for some quiver  $Q$  and some dimension vector  $\alpha$ . Show that  $K[\mathrm{Rep}(Q, \alpha)]$  is a polynomial ring of dimension  $\leq n - 1$  where  $n = \#Q_0$  is the number of vertices of  $Q$ .

**Exercise 5.** Find generators of the ring of semi-invariants for the quiver



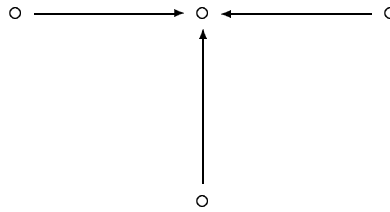
and the dimension vector

$$(n, n, n, \dots, n).$$

**Exercise 6.** Find generators of the ring of semi-invariants for the representations

$$\begin{array}{ccccc} K^n & \longrightarrow & K^{2n} & \longleftarrow & K^n \\ & & \uparrow & & \\ & & K^n & & \end{array}$$

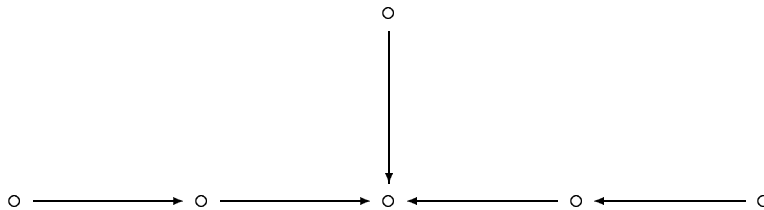
i.e., the quiver is



and the dimension vector is

$$\begin{array}{ccc} n & 2n & n \\ & n & \end{array}.$$

**Exercise 7.** Find the 5 generating semi-invariants for the quiver



and the dimension vector

$$\begin{array}{cccccc} & & 2n & & & \\ n & 2n & 3n & 2n & n & \cdot \end{array}$$