LECTURE 7:THE FUNDAMENTAL THEOREM'S OF INVARIANT THEORY

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In this lecture we will discuss the fundamental theorems in invariant theory for $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$. Proofs can be found for example in Claudio Procesi, *A primer in Invariant Theory.* These Theorems will be useful for understanding rings of (semi)-invariants for quiver representations. In particular, in the end we will prove a result of Kac that the rings of semiinvariants $\mathrm{SI}(Q,\alpha)$ and $\mathrm{SI}(s_x, s_x\alpha)$ are isomorphic if x is a source or a sink.

1. The FFT and SFT for GL(V)

Let us first introduce the first fundamental theorem (FFT) for $\mathrm{GL}(V)$. Let us consider the $\mathrm{GL}(V)$-invariant polynomials on the vector space

$$V^r \oplus (V^\star)^s.$$

Using the pairing $V \times V^\star \to K$, it is easy to see that the function

$$(v_1, v_2, \dots, v_r, w_1, \dots, w_s) \mapsto (v_i, w_j)$$

is $\mathrm{GL}(V)$-invariant. This polynomial is symbolically denoted by (i,j).

Theorem 1 (FFT for $\mathrm{GL}(V)$)**.** *The ring of invariants*

$$K[V^m \oplus (V^\star)^n]$$

is generated by all (i,j), $1 \leq i \leq m$, $1 \leq j \leq n$.

The second fundamental Theorem states what the relations are between these generating invariants. Let $b_{i,j}$ be indeterminates, then we have a surjective ring homomorphism

$$\varphi : K[\{b_{i,j}\}] \to K[V^m \oplus (V^\star)^n]^{\mathrm{GL}(V)}$$

defined by

$$b_{i,j} \mapsto (i,j).$$

Theorem 2 (SFT for $\mathrm{GL}(V)$)**.** *The kernel of* φ *is generated by all*

$$\det \begin{pmatrix} b_{i_1,j_1} & b_{i_1,j_2} & \cdots & b_{i_1,j_{r+1}} \\ b_{i_2,j_1} & b_{i_2,j_2} & & b_{i_2,j_{r+1}} \\ \vdots & & \ddots & \vdots \\ b_{i_{r+1},j_1} & b_{i_{r+1},j_2} & \cdots & b_{i_{r+1},j_{r+1}} \end{pmatrix}$$

where $r = \dim(V)$ *and* $1 \leq i_1 < i_2 < \cdots < i_{r+1} \leq m$ *and* $1 \leq j_1 < j_2 < \cdots < j_{r+1} \leq n$.

In other words, the relations between the generators $b_{i,j}$ exacly say that the matrix

$$\begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & & \vdots \\ b_{m,1} & \cdots & b_{m,n} \end{pmatrix}$$

has rank at most r.

Recall that the affine variety whose coordinate ring is the invariant ring $K[X]^G$ (X an affine variety with a regular G-action) is denoted by $X/\!\!/G$. We can identify $V^m \cong V \otimes W^\star \cong Hom(W, V)$ and $(V^\star)^n \cong V^\star \otimes Z \cong Hom(V, Z)$ for some vector spaces W and Z of dimension m and n respectively. Now the first and second fundamental theorem can be formulated as follows:

Corollary 1. *We have*

$$\mathrm{Hom}(W, V) \times \mathrm{Hom}(V, Z)/\!\!/\,\mathrm{GL}(V) \cong \mathrm{Hom}^{(r)}(W, Z)$$

where $Hom^{(r)}(W, Z)$ *is the set of linear maps of rank at most* r. *The quotient map is given by*

$$(A, B) \in \mathrm{Hom}(W, V) \times \mathrm{Hom}(V, Z) \to BA \in \mathrm{Hom}^{(r)}(W, Z).$$

2. Grassmannian

Let $\bigwedge^r W$ be the r-th exterior power of W, i.e., the quotient space of $W^{\otimes r} = W \otimes W \otimes \cdot \otimes W$ by the subspace of all

$$w_1 \otimes w_2 \otimes \cdots \otimes w_r - \mathrm{sgn}(\sigma) w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \cdots \otimes w_r$$

with $w_1, \ldots, w_r \in W$, and σ a permuation where $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ. The image of $w_1 \otimes w_2 \otimes \cdots \otimes w_r$ in $\bigwedge^r W$ is denoted by

$$w_1 \wedge w_2 \wedge \cdots \wedge w_r.$$

If W has dimension n, then $\bigwedge^n W \cong K$ (and $\mathrm{GL}(W)$ acts on $\bigwedge^n W$ as multiplication by the determinant). We have a nondegenerate pairing

$$\bigwedge^r W \times \bigwedge^{n-r} W \to K$$

defined by

$$(v, w) \mapsto v \wedge w$$

and in this way the spaces $\bigwedge^r W$ and $\bigwedge^{n-r} W$ are dual to each other in a canonical way. Also $\bigwedge^r W$ and $\bigwedge^r (W^\star)$ are dual to each other in a canonical way using the pairing

$$\bigwedge^r W \times \bigwedge^r (W^\star) \to K$$

defined by

$$w_1 \wedge w_2 \wedge \cdots \wedge w_r, v_1 \wedge v_2 \wedge \cdots \wedge v_r \mapsto \det \begin{pmatrix} (v_1, w_1) & \cdots & (v_1, w_r) \\ \vdots & & \vdots \\ (v_r, w_1) & \cdots & (v_r, w_r) \end{pmatrix}.$$

(note that this indeed is well-defined because of the multilinearity and alternating propertyies of the determinant). This now allows us to identify $\bigwedge^r W$ and $\bigwedge^{n-r}(W^\star)$ because they are both dual to $\bigwedge^{n-r} W$.

Consider the set of all

$$w_1 \wedge w_2 \wedge \cdots \wedge w_r, \quad w_1, \dots, w_r \in W$$

inside $\bigwedge^r W$. This set is actually an affine subvariety of the vector space $\bigwedge^r W$ which we will call the **affine Grassmannian** and it will be denoted by $\widetilde{\mathrm{Gr}}(r, W)$. As usual we can construct the projective space $\mathbb{P}(\bigwedge^r W)$ by dividing $\bigwedge^r W \setminus \{0\}$ by the action $K^\star$. We can also divide the cone $\widetilde{\mathrm{Gr}}(r, W)\{0\}$ by the action of $K^\star$ to obtain a projective variety $\mathrm{Gr}(r, W)$ called the **Grassmannian**. This variety can be thought of as the set of r-dimensional subspaces of W. Indeed, if $V \subset W$ is an r-dimensional subspace, and $v_1, \dots, v_r$ is a basis of V, then this defines a projective point (which is independent of the basis choice of V)

$$[v_1 \wedge v_2 \wedge \cdots \wedge v_r]$$

in $\mathrm{Gr}(r, W)$.

The identification between $\bigwedge^r W$ and $\bigwedge^{n-r} W^\star$ induces an identification between $\widetilde{\mathrm{Gr}}(r, W)$ and $\widetilde{\mathrm{Gr}}(n - r, W^\star)$ and an identification between $\mathrm{Gr}(r, W)$ and $\mathrm{Gr}(n - r, W^\star)$. The latter identification can also be understood as follows: If $V \subset W$ is an r-dimensional subspace, then dualizing gives us a surjective map $W^\star \to V^\star$. The kernel is an $n - r$-dimensional subspace of $W^\star$. This gives a bijection (and in fact an isomorphism of algebraic varieties) between $\mathrm{Gr}(r, W)$ and $\mathrm{Gr}(n - r, W^\star)$.

Let us describe the affine coordinate ring $\widetilde{\mathrm{Gr}}(r, W^\star) \subset \bigwedge^r(W^\star) = (\bigwedge^r W)\star$. Elements on $\bigwedge^r W$ are linear functions on $\bigwedge^r(W^\star)$ (and on $\widetilde{\mathrm{Gr}}(r, W^\star)$). Choose a basis $w_1, \dots, w_n$ of W and denote

$$p_{i_1, i_2, \dots, i_r} = w_{i_1} \wedge w_{i_2} \wedge \cdots \wedge w_{i_r}.$$

for all $1 \leq i_1 < \cdots < i_r \leq n$. These linear functions are called **Plücker coordinates**. The coordinate ring of $\bigwedge^r(W^\star)$ is the polynomial ring in all

$$p_{i_1, \dots, i_r}, \quad 1 \leq i_1 < i_2 < \cdots < i_r \leq n.$$

Let us also define

$$p_{i_{\sigma(1)}, \dots, i_{\sigma(r)}} = \mathrm{sgn}(\sigma) p_{i_1, \dots, i_r}$$

for all permutations σ. The vanishing ideal of the subvariety $\widetilde{\mathrm{Gr}}(r, W^\star)$ is generated by the **Grassmann-Plücker relations**:

$$\sum_{k=1}^{r+1} (-1)^k p_{i_1,i_2,\dots,i_{r-1},j_k} p_{j_1,\dots,\widehat{j_k},\dots,j_{r+1}}$$

for all $i_1, \dots, i_{r-1}, j_1, \dots, j_{r+1}$.

The Grassmannian will be used in the future but also will pop up if we discuss the fundamental theorems for $\mathrm{SL}(V)$.

3. The fundamental Theorems for SL(V)

Let V be an r-dimensional vector space. We would like to understand the invariant ring

$$K[V^n]^{\mathrm{SL}(V)}.$$

For any $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ we define a function $V^n \to K$ by

$$(v_1, \dots, v_n) \mapsto \det(v_{i_1}\ v_{i_2}\ \cdots\ v_{i_r}).$$

This invariant function will symbolically be denoted by the bracket $[i_1\ i_2\ \cdots\ i_r]$.

Theorem 3 (FFT for $\mathrm{SL}(V)$)**.** *The invariant ring*

$$K[V^n]^{\mathrm{SL}(V)}$$

is generated by all $[i_1\ i_2\ \cdots\ i_r]$ *with* $1 \leq i_1 < i_2 < \cdots < i_r \leq n$.

We can identify V^n with $V \otimes W$ where W is an n-dimensional space. Consider the morphism

$$\pi : V \otimes W \to \wedge^r W$$

defined by

$$\sum_{i=1^r} v_i \otimes w_i \mapsto \det(v_1\ v_2\ \cdots\ v_r) w_1 \wedge w_2 \wedge \cdots \wedge w_r.$$

Note that this is a well defined morphism whose image is exactly $\widetilde{\mathrm{Gr}}(r, W)$. The pullback functions $\pi^\star p$ of all Plücker coordinates p, give exactly the generating invariants for $K[V \otimes W]^{\mathrm{SL}(V)}$.

Corollary 2. *We have*

$$V \otimes W /\!/ \mathrm{SL}(V) \cong \widetilde{\mathrm{Gr}}(r, W)$$

The description in the previous section of the coordinate ring of $\widetilde{\mathrm{Gr}}(r, W)$ can be thought of as the second fundamental Theorem for $\mathrm{SL}(V)$.

4. Kac' Theorem on Semiinvariants

We will now show a theorem on the ring of semi-invariants

Theorem 4 (Kac). *If x is a sink or a source for a quiver Q and α is a dimension vector with $(s_x\alpha)(x) \geq 0$, then*

$$\mathrm{SI}(Q,\alpha) \cong \mathrm{SI}(s_xQ, s_x\alpha)$$

Proof. We will prove the case that x is a sink for Q (it will be a source for s_xQ). Put $r = \alpha(x)$ and $n = \sum_{ha=x}\alpha(ta)$. Note that $s_x(\alpha)(x) = n - r$. Put $V = K^r$, $V' = K^{n-r}$ and $W = \bigoplus_{ha=x} K^{\alpha(ta)} \cong K^n$. Also define $Z = \bigoplus_{ha\neq x}\mathrm{Hom}(K^{\alpha(ta)}, K^{\alpha(ha)})$, and $G = \prod_{y\neq x}\mathrm{GL}(\alpha(y))$. Now we have

$$\begin{aligned}\mathrm{SI}(Q,\alpha) &= K[\mathrm{Rep}(Q,\alpha)]^{\mathrm{SL}(\alpha)} = K[Z\times \mathrm{Hom}(W,V)]^{G\times \mathrm{SL}(V)} = \\ &= (K[Z]\otimes K[\mathrm{Hom}(W,V)]^{\mathrm{SL}(V)})^G = (K[Z]\otimes K[\widetilde{\mathrm{Gr}}(r,W^\star)])^G\end{aligned}$$

and

$$\begin{aligned}\mathrm{SI}(s_xQ,s_x\alpha) &= K[\mathrm{Rep}(s_xQ,s_x\alpha)]^{\mathrm{SL}(\alpha)} = K[Z\times \mathrm{Hom}(V',W)]^{G\times \mathrm{SL}(V')} = \\ &= (K[Z]\otimes K[\mathrm{Hom}(V',W)]^{\mathrm{SL}(V')})^G = (K[Z]\otimes K[\widetilde{\mathrm{Gr}}(n-r,W)])^G.\end{aligned}$$

Since the varieties $\widetilde{\mathrm{Gr}}(r,W^\star)$ and $\widetilde{\mathrm{Gr}}(n-r,W)$ are isomorphic and the isomorphy respects the action of G, it follows immediately that $\mathrm{SI}(Q,\alpha)$ and $\mathrm{SI}(s_xQ,s_x\alpha)$ are isomorphic. □

For a quiver Q without oriented cycles, let us define how the reflection s_x acts on a weight σ. We define

$$\begin{aligned}(s_x\sigma)(x) &= -\sigma(x) && \text{and}\\ (s_x\sigma)(y) &= \sigma(y)+\sigma(x)b_{x,y} && \text{if } y\neq x,\end{aligned}$$

where $b_{x,y}$ is the number of arrows between x and y.

Remark 1. Suppose that x is a sink or a source of the quiver, then the isomorphism

$$\varphi : \mathrm{SI}(Q,\alpha) \to \mathrm{SI}(s_xQ,s_x\alpha)$$

induces an isomorphism for each weight

$$\varphi_\sigma : \mathrm{SI}(Q,\alpha)_\sigma \to \mathrm{SI}(s_xQ,s_x\alpha)_{s_x\sigma}$$

5. Exercises

Exercise 1. Consider the ring

$$K[\mathrm{Rep}(Q,\alpha)] = K[\bigoplus_{a\in Q_1}\mathrm{Hom}(K^{\alpha(ta)},K^{\alpha(ha)})] = \bigotimes_{a\in Q_1} K[\mathrm{Hom}(K^{\alpha(ta)},K^{\alpha(ha)})].$$

This ring is multigraded, namely for each $K[\mathrm{Hom}(K^{\alpha(ta)},K^{\alpha(ha)})]$ we have the usual degree function. Suppose that $f \in K[\mathrm{Rep}(Q,\alpha)]$ is multihomogeneous. Then its multidegree is a function $d : Q_1 \to \mathbb{N}$.

(a). Suppose that $f \in \mathrm{SI}(Q,\alpha) \subset K[\mathrm{Rep}(Q,\alpha)]$ is multihomogeneous with multidegree $d : Q_1 \to \mathbb{N}$ which is invariant under the action of $\mathrm{SL}(\alpha)$. Show that f is a semiinvariant of weight σ where $\sigma : Q_0 \to \mathbb{Z}$ is given by

$$\sigma(x) = \frac{\sum_{\substack{a \in Q_1 \\ ta=x}} d(a) - \sum_{\substack{a\in Q_1 \\ ha=x}} d(a)}{\alpha(x)}.$$

(Hint: look at the action of $\lambda\,\mathrm{id}_{\alpha(x)} \in \mathrm{GL}(\alpha(x))$ for every vertex x.)

(b). Suppose that x is a sink or a source of Q. If $f \in \mathrm{SI}(Q,\alpha)$ is a semiinvariant of degree σ, show that under the isomorphy

$$\varphi : \mathrm{SI}(Q,\alpha) \to \varphi(s_xQ, s_x\alpha)$$

$\varphi(f)$ is a semiinvariant of weight $s_x\sigma$. (You may assume that f is also multihomogeneous. Note that the multidegree remains unchanged under the isomorphy.)

Exercise 2. Suppose that x is a source or a sink of the quiver Q. Show that if $s_x\alpha(x) < 0$ and f is a semiinvariant of weight σ, then $\sigma(x) = 0$. (Hint: Define $A(y) = \mathrm{id}_{\alpha(y)}$ for $y \neq x$ and take $A(x)$ in such way that $\det(A(x))$ is not a root of unity and $A = \{A(y) \mid y \in Q_0\} \in \mathrm{GL}(\alpha)$ acts trivially on V.)

Exercise 3. Let Q be the the quiver

$$\circ \rightrightarrows \circ$$

and α be the dimension vector (n,m). Using reflections and Kac' Theorem on semiinvariant, show that

$$\mathrm{SI}(Q,\alpha) \neq K$$

if and only if $n = m$ or

$$\frac{|n-m|}{\gcd(n,m)} = 1$$

Exercise 4. Using reflections, find the ring of semiinvariants $\mathrm{SI}(Q,\alpha)$ where Q is the quiver

$$\circ \longrightarrow \circ \rightrightarrows \circ$$

and $\alpha = (4,5,3)$ (you don't have to find explicit formula's for the semiinvariants). What are the weights of the generating semi-invariants.