LECTURE 8: SCHUR FUNCTORS

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In this Lecture, we will discuss the irreducible polynomial representations of GL(V), and we will apply this theory to semiinvariants of quiver representations. Proofs of results about the representations theory of GL(V) can be found in William Fulton, Young Tableaux, London Mathematical Society Student Texts.

1. Partitions

Definition 1. A partition of \( n \) is a tuple \( \lambda = (\lambda_1, \ldots, \lambda_r) \) of positive integers with \( \lambda_1 \geq \lambda_2 \geq \cdots \lambda_r \geq 1 \) and \( |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_r = n \).

To each partition we can associate a Young diagram, for example, the Young diagram of \( \lambda = (5, 4, 2, 1) \) is

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

If we mirror the diagram of a partition \( \lambda \) in the diagonal, then we get the Young diagram of the conjugate partition \( \lambda' \):

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

We let \( V^{\times \lambda} \cong V^{[\lambda]} \) be the set of all maps from boxes in the Young diagram to \( V \), i.e., we can think of \( V^{\times \lambda} \) as the set of Young diagrams, whose boxes are filled with elements of \( V \), for example

\[
\begin{array}{cccccc}
a & b & c & d & e \\
f & g & h \\
i \\
\end{array}
\]
We need the notion of an exchange. Suppose that \( v \in V^{\times \lambda} \). An exchange is when we take two columns of \( \lambda \) and take a subset of each column (with the same cardinality) and we exchange the elements of the subsets (keeping the order in which they appear in each column intact). For example

\[
\begin{array}{cccc}
\text{a} & \text{b} & \text{c} & \text{d} \\
\text{e} & \text{f} & \text{g} & \\
\text{h} & \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
\text{c} & \text{b} & \text{a} & \text{d} \\
\text{e} & \text{f} & \text{h} & \\
\text{g} & \\
\end{array}
\]

2. Construction of the irreducible polynomial representations of \( \text{GL}(V) \)

Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \) be a partition and let \( \mu = (\mu_1, \ldots, \mu_s) \) be the dual partition. We define a map

\[
\wedge : V^{\times \lambda} \rightarrow \bigwedge^{\mu_1} V \otimes \bigwedge^{\mu_2} V \otimes \cdots \otimes \bigwedge^{\mu_s} V
\]

by taking the antisymmetric product over all columns and then taking the tensor product of those. For example, the element

\[
\begin{array}{cccc}
\text{a} & \text{b} & \text{c} & \text{d} \\
\text{f} & \text{g} & \text{h} \\
\text{i} & \\
\end{array}
\]

maps to \((a \wedge f \wedge i) \otimes (b \wedge g) \otimes (c \wedge h) \otimes d \otimes e \in \bigwedge^3(V) \otimes \bigwedge^2(V) \otimes \bigwedge^2(V) \otimes V\).

We define

\[
S^\lambda(V) = \bigwedge^{\mu_1} V \otimes \bigwedge^{\mu_2} V \otimes \cdots \otimes \bigwedge^{\mu_s} V / Q^\lambda(V)
\]

where \( Q^\lambda(V) \) is a subspace of \( \bigotimes_i \bigwedge^{\mu_i}(V) \) spanned by all \( \wedge v - \sum w \wedge w \) where the sum is over all \( w \) which are obtained by an exchange of two given columns and a given subset in the right column.

**Example 1.** We have

\[
S^{(2,1)}(V) = \bigwedge^2 V \otimes V / Q^{(2,1)}V
\]

where \( Q^{(2,1)}(V) \) is the space spanned by all

\[
\begin{align*}
\wedge & \begin{array}{ccc}
\text{v} & \text{w} \\
\text{z} & \\
\end{array} = \quad \wedge & \begin{array}{ccc}
\text{w} & \text{v} \\
\text{z} & \\
\end{array} = \quad \wedge & \begin{array}{ccc}
\text{v} & \text{z} \\
\text{w} & \\
\end{array} = \\
\end{align*}
\]

\[
= (v \wedge z) \otimes w - (w \wedge z) \otimes v - (v \wedge w) \otimes z,
\]

with \( v, w, z \in V \).
**Example 2.** If we take the partition $\lambda = (n)$, then we get
\[ S^{(n)} V \cong S^n V, \]
the usual symmetric power of $V$. We also can view $S^n (V)$ as the set of homogeneous polynomials on $V^*$ of degree $n$. For a vector space, the coordinate ring $K[V]$ can be identified with $SV^* = \bigoplus_{n \geq 0} S^n V^*$, the symmetric algebra on $V^*$.

**Example 3.** If we take the partition $\lambda = (1, 1, \ldots, 1) = (1^n)$, then
\[ S^{(1^n)} V = \bigwedge^n V. \]

**Definition 2.** Let $\lambda$ be a partition. A tableau is a filling of the Young diagram with positive integers which is
(a). weakly increasing along each row and
(b). strictly increasing along each column.

For example,

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & 3 \\
3 & 4 \\
5
\end{array}
\]

Let us choose a basis $e_1, \ldots, e_m$ of $V$. To each tableau $T$ of a partition $\lambda$ with entries in $\{1, 2, \ldots, m\}$ we can associate an element $e_T \in S^\lambda (V)$ as in the following example:

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & 3 \\
3 & 4 \\
5
\end{array}
\]

\[ e_T = (e_1 \wedge e_2 \wedge e_3 \wedge e_5) \otimes (e_1 \wedge e_2 \wedge e_4) \otimes (e_1 \wedge e_2) \otimes (e_2 \wedge e_3) \otimes e_2 + Q^\lambda (V) \in S^\lambda (V). \]

**Theorem 1.** The elements
\[ \{ e_T \mid T \text{ is a tableau on } \lambda \text{ with entries in } \{1, 2, \ldots, m\} \}. \]
form a basis of $S^\lambda (V)$.

**Proof.** See Theorem 1 in §8.1 in Fulton’s Book on Young Tableaux. \qed

Note that from this theorem follows that
\[ S^\lambda (V) = 0 \iff \lambda \text{ has more than } m = \dim (V) \text{ parts}. \]
dim $S^\lambda(V) = 1 \iff \lambda = (k^m)$ for some $k$.

If $W$ is a regular representation of $GL(V)$, then the action is given by a morphism $\varphi : GL(V) \to GL(W)$. We say that the representation is polynomial if the morphism $\varphi$ extends to a morphism $End(V) \to End(W)$. This means that for every polynomial function $f$ on $End(W)$, $\varphi^* f := f \circ \varphi$ is a polynomial function on $End(V)$. Since $K[GL(V)] = K[End(V)]_{det}$ is the localization, we see that if the representation is not polynomial, we still have $\varphi^* f = g/\det^k$ for some polynomial function $g$ and some positive integer $k$.

**Theorem 2.** The irreducible polynomial representations of $GL(V)$ are exactly all

$$S^\lambda(V)$$

where $\lambda$ is a partition with at most $m := \dim(V)$ parts.

**Proof.** See Theorem 2 in § 8.2 in Fulton, *Young Tableaux*. □

For any integer $k$ we have a one dimensional representation of $GL(V)$ defined by

$$A \cdot v \mapsto \det(A)^k v, \quad A \in GL(V), v \in K.$$ 

This representation we will denote by $\det(V)^k$. If $W$ is any representation, we can “twist” the action to $W \otimes \det(V)^k$. Also note that $W \otimes \det(V)^k \otimes \det(V)^l \cong W \otimes \det(V)^{k+l}$. If $W$ is any irreducible representation of $GL(V)$, then for an integer $k$ which is large enough we will have that $W \otimes \det(V)^k$ is an irreducible polynomial representation, and therefore equal to $S^\lambda(V)$ for some $\lambda$.

If $W = S^\lambda(V)$ is already an irreducible polynomial representation, then $W \otimes \det(V)^k$ is also a polynomial irreducible representation, so $W \otimes \det(V)^k \cong S^\mu(V)$ for some $\mu$. What is $\mu$? Well, it turns out that if $\lambda = (\lambda_1, \ldots, \lambda_r)$, then

$$S^\lambda(V) \otimes \det(V)^k = S^{(\lambda_1+k, \lambda_2+k, \ldots, \lambda_m+k)}(V)$$

where we take $\lambda_i = 0$ for $i > r$.

**Corollary 1.** The irreducible representations of $GL(V)$ are exactly all

$$S^\lambda(V) \otimes \det(Z)^k$$

where $k \in \mathbb{Z}$ and $\lambda$ is a partition with at most $m - 1$ parts.

Note that for distinct $k$’s, the spaces $S^\lambda(V) \otimes \det(Z)^k$ are all isomorphic as representations of $SL(V)$.

**Proposition 1.** The set of irreducible representations of $SL(V)$ are exactly all $S^\lambda(V)$ with $\lambda$ a partition with at most $m - 1$ parts where $m := \dim(V)$.

If $W = S^\lambda(V)$ is an irreducible representation of $SL(V)$, then $W^*$ is also an irreducible representation of $SL(V)$, so $W^* = S^\mu(V)$ for some partition $\mu$. What is $\mu$? Well, if we turn the Young diagram of $\mu$ around over 180°, then $\lambda$ and $\mu$ fit together to make an $m \times k$ square for some $k$:
In other words, by Schur’s Lemma, \((S^\lambda(V) \otimes S^\mu(V))^{SL(V)} \cong \text{Hom}((S^\lambda(V))^*, S^\mu(V))^{SL(V)} \neq 0\), if and only if \(S^\mu(V) \cong (S^\lambda(V))^*\) as \(SL(V)\)-representations. So \((S^\lambda(V) \otimes S^\mu(V))^{SL(V)} \neq 0\) if and only if \(\lambda\) and \(\mu\) fit together to make an \(m \times k\) square for some \(k\) where \(m := \text{dim}(V)\). In that case, \((S^\lambda(V) \otimes S^\mu(V))^{SL(V)}\) is one-dimensional.

The map \(V \mapsto S^\lambda(V)\) can actually be extended to a functor from the category of finite dimensional vector spaces to itself. These functors are called Schur functors.

3. Computations with Schur functors

Let us first list some properties of Schur functors. Since \(S^\lambda(V) \otimes S^\mu(V)\) is again a polynomial representation of \(GL(V)\), it decomposes again into irreducible polynomial representations:

\[
S^\lambda(V) \otimes S^\mu(V) = \bigotimes_{\nu} S^\nu(V)^{c_{\lambda,\mu}^\nu}.
\]

For some nonnegative integers \(c_{\lambda,\mu}^\nu\). These numbers are called Littlewood-Richardson coefficients. By considering the action of \(\lambda \text{id}\), it is clear that the sum is over partitions \(\nu\) with \(|\nu| = |\lambda| + |\mu|\). Note that \(S^\nu(V \oplus W)\) is a polynomial representation of \(GL(V) \times GL(W)\). Irreducible representations of \(GL(V) \times GL(W)\) are just tensor products of irreducible representations of \(V\) with irreducible representations of \(GL(W)\). We have a decomposition

\[
S^\nu(V \oplus W) = \bigoplus_{\lambda,\mu} (S^\lambda(V) \otimes S^\mu(W))^{c_{\lambda,\mu}^\nu}
\]

where the multiplicities turn out to be again the Littlewood-Richardson coefficients. The sum is over all \(\lambda, \mu\) with \(|\lambda| + |\mu| = |\nu|\). Special cases are when \(\lambda = (n)\) or \(\lambda = (1^n)\):

\[
S^n(V \oplus W) \cong \bigoplus_{a+b=n} S^a(V) \otimes S^b(W),
\]

\[
\wedge^n(V \oplus W) \cong \bigoplus_{a+b=n} \wedge^a(V) \otimes \wedge^b(V).
\]
The following Cauchy-formula will also be useful for computations
\[ S^n(V \otimes W) \cong \bigoplus_{|\lambda|=n} S^\lambda(V) \otimes S^\lambda(W). \]
its counter part is
\[ \bigwedge^n(V \otimes W) \cong \bigoplus_{|\lambda|=n} S^\lambda(V) \otimes S^\lambda(W) \]
where \( \lambda' \) is the conjugate partition. These two formula’s generalize to
\[ S^n(V \otimes W) \cong \bigoplus_{|\lambda|=|\mu|=n} (S^\lambda(V) \otimes S^\mu(V))^a_{\lambda,\mu}. \]
There is an explicit description of the multiplicities \( a^\alpha_{\lambda,\mu} \). The irreducible representations of the symmetric group \( S_n \) are also parametrized by partition of \( n \) (we won’t need this and therefore we will not discuss it here). The coefficients \( a^\alpha_{\lambda,\mu} \) are now tensor product multiplicities for the symmetric group \( S_n \).

**Example 4.** Let us consider the quiver \( Q \) given by
\[ \circ \quad \rightarrow \quad \circ \]
and the dimension vector \( \alpha = (m, n) \). Let \( V = K^m \) and \( W = K^n \). Then
\[ \text{SI}(Q, \alpha) = K[\text{Hom}(V, W)]^{SL(V) \times SL(W)} = K[V^* \otimes W]^{SL(V) \times SL(W)} = S(V \otimes W^*)^{SL(V) \times SL(W)} = \bigoplus_{\lambda} (S^\lambda(V) \otimes S^\lambda(W^*))^{SL(V) \times SL(W)} = \]
Now \( S^\lambda(V)^{SL(V)} \neq 0 \) if and only if \( \lambda = (k^m) \) for some \( k \) and \( S^\lambda(W^*)^{SL(W)} \neq 0 \) if and only if \( \lambda = (l^n) \) for some \( l \). It follows that if \( n \neq m \) then there are only trivial semi-invariants, i.e., \( \text{SI}(Q, \alpha) = K \). If \( n = m \), then for every \( \lambda = (k^n) \) the space \( S^\lambda(V)^{SL(V) \otimes S^\lambda(W^*)^{SL(W)}} \) is one-dimensional. The weight of the semi-invariant spanning this space is \((k, -k)\). Note that
\[ A \in \text{Hom}(V, W) \mapsto \det(A)^k \]
is a semi-invariant of weight \((k, -k)\) so it therefore spans \( \text{SI}(Q, (n, n))_{(k, -k)} \). We also see that
\[ \text{SI}(Q, (n, n)) = K[\det], \]
i.e., the ring of semi-invariants is generated by the determinant.
Example 5. Consider the quiver \( Q \):

\[
\circ \rightarrow \circ
\]

and the dimension vector \( \alpha = (m, n) \). Let \( V = K^m \) and \( W = K^n \). Then

\[
\text{SI}(Q, \alpha) = K[\text{Hom}(V, W) \oplus \text{Hom}(V, W)]^{\text{SL}(V) \times \text{SL}(W)} = S(V \otimes W^* \otimes V \otimes W^*)^{\text{SL}(V) \times \text{SL}(W)} =
\]

\[
= (S(V \otimes W^*) \otimes S(V \otimes W^*))^{\text{SL}(V) \times \text{SL}(W)} =
\]

\[
= \bigoplus_{\lambda, \mu} (S^\lambda(V) \otimes S^\mu(W^*) \otimes S^\mu(V) \otimes S^\mu(W^*))^{\text{SL}(V) \times \text{SL}(W)} =
\]

\[
= \bigoplus_{\lambda, \mu} (S^\lambda(V) \otimes S^\mu(V))^{\text{SL}(V)} \otimes (S^\lambda(W^*) \otimes S^\mu(W^*))^{\text{SL}(W)}.
\]

If \((S^\lambda(V) \otimes S^\mu(V))^{\text{SL}(V)} \neq 0\) then \( \lambda \) and \( \mu \) fit together to make an \( m \times k \) square for some \( k \). If \((S^\lambda(W^*) \otimes S^\mu(W^*))^{\text{SL}(W)} \neq 0\) then \( \lambda \) and \( \mu \) must also fit together to make an \( n \times l \) square for some \( l \). If

\[
(S^\lambda(V) \otimes S^\mu(V))^{\text{SL}(V)} \otimes (S^\lambda(W^*) \otimes S^\mu(W^*))^{\text{SL}(W)} \neq 0
\]

then it consists of semi-invariants of weight \((k, -l)\) (and \( km = |\lambda| + |\mu| = ln \)).

We distinguish two cases:

Case 1: \( n = m \). If \( \lambda \) and \( \mu \) fit together to make an \( m \times k \) rectangle then

\[
S(S^\lambda(V) \otimes S^\mu(V))^{\text{SL}(V)} \otimes (S^\lambda(W^*) \otimes S^\mu(W^*))^{\text{SL}(W)}
\]

is a one-dimensional space of semiinvariants of weight \((k, -k)\). Note that there are exactly \( \binom{m+k}{k} \) possible partitions \( \lambda \) fitting in an \( m \times k \) rectangle. This means that there are exactly \( \binom{m+k}{k} \) linearly independent semiinvariants of weight \((k, -k)\).

This shows that

\[
\dim \text{SI}(Q, (n, n))_{(k, -k)} = \binom{m+k}{k}.
\]

The coefficients of

\[
(A, B) \in \text{Hom}(V, W)^2 \mapsto \det(A + tB) = f_0(A, B) + f_1(A, B)t + \cdots + f_n(A, B)t^n
\]

as a polynomial in \( t \) give \( n + 1 \) linearly independent semiinvariants \( f_0, f_1, \ldots, f_n \) of weight \((1, -1)\). These semi-invariants are algebraically independent. There are exactly \( \binom{m+k}{k} \) monomials in \( f_0, \ldots, f_n \) of degree \( k \) (which are semi-invariants of weight \((k, -k)\)). It follows that the ring of semiinvariants is the polynomial ring

\[
K[f_0, f_1, \ldots, f_n].
\]

Case 2: \( n \neq m \). \( \lambda \) and \( \mu \) fit together to form an \( n \times k \) and an \( m \times l \) rectangle. Then \( \lambda \) and \( \mu \) must be “staircases” as shown below:
Suppose \( n > m \). If the staircase consists of \( a \times b \) “blocks”, then \( n = (r + 1)a \), \( m = ra \), \( k = rb \) and \( l = (r + 1)b \) where \( r \) is the number of steps. Note that \( n - m = a = \text{lcm}(n, m) \). A similar description we get if \( n < m \). We conclude that if \( |n - m|/\text{lcm}(n, m) > 1 \) then \( \text{SI}(Q, \alpha) = K \). If \( |n - m| = \gcd(m, n) \), then there exists exactly one semi-invariant of weight \( (nb/\text{lcm}(n, m), -mb/\text{lcm}(n, m)) \) (up to scalar multiples) for all \( b \geq 0 \). In this case \( \text{SI}(Q, \alpha) \) is the polynomial ring in one variable.

The following example will be quite important.
4. The triple flag quiver

Let \( Q = T_{p,q,r} \) be the quiver

\[
1 \rightarrow 2 \rightarrow \cdots \rightarrow p-1 \rightarrow p \rightarrow r-1 \rightarrow \cdots \rightarrow T
\]

We will use the convention \( q = q' = p \). Let \( \alpha \) be a dimension vector and \( \sigma \) be a weight. We will assume that \( \sigma(\alpha) = 0, \alpha(1) \leq \alpha(2) \leq \cdots \alpha(p), \alpha(\bar{1}) \leq \cdots \leq \alpha(q), \alpha(\bar{\bar{1}}) \leq \cdots \leq \alpha(\bar{\bar{q}}) \) and \( \sigma(i) > 0 \) for all \( i < p \), \( \sigma(i) > 0 \) for all \( i < q \) and \( \sigma(i) > 0 \) for all \( i < r \) and \( \sigma(p) < 0 \).

**Proposition 2.** We have an isomorphism

\[
\text{SI}(Q, \alpha)_\sigma \cong (S^\mu U \otimes S^\nu U \otimes S^\omega U)^{\text{SL}(U)}
\]

where \( U \) is a vector space of dimension \( \alpha(p) \) and the conjugate partitions \( \mu', \nu', \omega' \) are given by

\[
\mu' = (\alpha(p-1)^{\sigma(p-1)}, \ldots, \alpha(1)^{\sigma(1)}).
\]

\[
\nu' = (\alpha(q-1)^{\sigma(q-1)}, \ldots, \alpha(1)^{\sigma(1)})
\]

\[
\omega' = (\alpha(r-1)^{\sigma(r-1)}, \ldots, \alpha(\bar{\bar{1}})^{\sigma(\bar{\bar{1}})}).
\]
Proof. Let us define $V_i = K^\alpha(i)$, and the same for $\overline{V}_i$ and $\overline{V}_i$ for all $i$.

$$\text{SI}(Q, \alpha) = K \left( \bigoplus_{i=1}^{p-1} \text{Hom}(V_i, V_{i+1}) \bigoplus_{i=1}^{q-1} \text{Hom}((\overline{V}_i, V_{i+1}) \bigoplus_{i=1}^{r-1} \text{Hom}(\overline{V}_i, V_{i+1})) \right)^{\text{SL}(\alpha)}$$

$$= \left( \bigotimes_{i=1}^{p-1} S(V_i \otimes V_i^*) \bigotimes_{i=1}^{q-1} S((\overline{V}_i \otimes V_i^*) \bigotimes_{i=1}^{r-1} S(\overline{V}_i \otimes V_i^*)) \right)^{\text{SL}(\alpha)}$$

By Cauchy’s formula, this is equal to

$$\bigoplus_{\lambda(\overline{i}), \lambda(\overline{q}), \lambda(\overline{r})} \left( S^{\lambda(\overline{1})} V_i \otimes S^{\lambda(\overline{2})} V_i^* \bigotimes_{i=1}^{q-1} S^{\lambda(\overline{3})} V_i \otimes S^{\lambda(\overline{4})} V_i^* \bigotimes_{i=1}^{r-1} S^{\lambda(\overline{5})} V_i \otimes S^{\lambda(\overline{6})} V_i^* \right)^{\text{SL}(\overline{V}) \otimes \text{SL}(\overline{V}) \otimes \text{SL}(\overline{V})}$$

where for each $i$, $\lambda(i)$ runs through all partitions, and the same for $\lambda(\overline{i})$ and $\lambda(\overline{i})$.

Note that $\text{SL}(\alpha) = \prod_{i=1}^{p-1} \text{SL}(V_i) \times \prod_{i=1}^{q-1} \text{SL}(V_i) \times \prod_{i=1}^{r-1} \text{SL}(V_i)$.

Sorting the terms by vector space, the previous expression is equal to

$$\bigoplus_{\lambda(\overline{i}), \lambda(\overline{q}), \lambda(\overline{r})} (S^{\lambda(\overline{1})} V_i)_{\text{SL}(V_i)} \bigotimes (S^{\lambda(\overline{2})} V_i)_{\text{SL}(V_i)} \bigotimes (S^{\lambda(\overline{3})} V_i)_{\text{SL}(V_i)} \bigotimes (S^{\lambda(\overline{4})} V_i)_{\text{SL}(V_i)} \bigotimes (S^{\lambda(\overline{5})} V_i)_{\text{SL}(V_i)} \bigotimes (S^{\lambda(\overline{6})} V_i)_{\text{SL}(V_i)}$$

Inside this (rather complicated) sum we will look for semi-invariants of weight $\sigma$. If $(S^{\lambda(\overline{1})} V_i)_{\text{SL}(V_i)} \neq 0$, then the partition $\lambda(1)$ must be a square partition with exactly $\lambda(1)$ rows. The number of columns in $\lambda(1)$ should be $\sigma(1)$, where $\sigma$ is the weight of semi-invariants we are looking at. In other words, $\lambda(1) = (\sigma(1)^{\alpha(1)})$ and $\lambda(1) = (\sigma(1)^{\sigma(1)})$ where $\lambda(1)$ is the conjugate partition of $\lambda(1)$. Also notice that for this $\lambda(1)$ we have $\dim(S^{\lambda(\overline{1})} V_i)_{\text{SL}(V_i)} = 1$.

$$\lambda(1):$$

$$\alpha(1)$$

$$\sigma(1)$$

In order for $(S^{\lambda(\overline{1})} V_i^* \otimes S^{\lambda(\overline{2})} V_i)_{\text{SL}(V_i)}$ to be nonzero, $\lambda(1)$ and $\lambda(2)$ must be the same, except that $\lambda(2)$ may have some extra columns of size $\alpha(2)$. In fact $\lambda(2)$ is equal to $\lambda(1)$ with $\sigma(2)$ columns of size $\alpha(2)$ added. So $\lambda(2) = (\alpha(2)^{\sigma(2)}, \alpha(1)^{\sigma(1)})$. For this $\lambda(1)$ and $\lambda(2)$ we have $\dim(S^{\lambda(\overline{1})} V_i^* \otimes S^{\lambda(\overline{2})} V_i)_{\text{SL}(V_i)} = 1$. 
Continuing this reasoning, we see that $\lambda(3)$ is equal to $\lambda(2)$ with $\sigma(3)$ columns of size $\alpha(3)$ added, so $\lambda(3)' = (\alpha(3)^{\sigma(3)}, \alpha(2)^{\sigma(2)}, \alpha(1)^{\sigma(1)})$ where $\lambda(3)'$ is the conjugate partition of $\lambda(3)$, etc., and finally we get

$$\mu := \lambda(p - 1) = (\alpha(p - 1)^{\sigma(p-1)}, \ldots, \alpha(1)^{\sigma(1)}).$$

Similarly we obtain

$$\nu := \lambda(q - 1) = (\alpha(q - 1)^{\sigma(q-1)}, \ldots, \alpha(1)^{\sigma(1)}).$$

and

$$\omega := \lambda(r - 1) = (\alpha(r - 1)^{\sigma(r-1)}, \ldots, \alpha(1)^{\sigma(1)}).$$

If we put $U = V_p^*$, then the space of semi-invariants of weight $\sigma$ is isomorphic to

$$(S^\mu U \otimes S^\nu U \otimes S^\omega U)^{\text{SL}(U)}.$$

We also have a variation on the previous proposition. Instead of studying the quiver $Q = T_{p,q,r}$, we give the third arm the opposite orientation, i.e., we consider
the quiver $T_{p,q}^r$ given by

\[
\begin{array}{cccccccc}
1 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & p-1 & \rightarrow & r-1 & \rightarrow & \cdots & \rightarrow & 1 \\
\downarrow & & & & & & & & & & & & q-1 \\
\uparrow & & & & & & & & & & & & \vdots \\
\uparrow & & & & & & & & & & & & 2 \\
\uparrow & & & & & & & & & & & & 1
\end{array}
\]

Let us assume that $\alpha$ is a dimension vector which is weakly increasing along arms. Assume that $\sigma$ is a weight with $\sigma(i) \geq 0$ for all $i < p$, $\sigma(\tau) \geq 0$ for all $i < q$ and $\sigma(\overline{r}) \leq 0$ for $i \leq r$.

**Proposition 3.** We have

$$\text{SI}(T_{p,q}^r, \alpha)_\sigma \cong (S^\mu U \otimes S^\nu U \otimes (S^\omega U)^*)^{GL(U)}.$$  

where the conjugate partitions $\mu'$, $\nu'$ and $\omega'$ are given by

- $\mu' = (\alpha(p-1)^{\sigma(p-1)}, \ldots, \alpha(1)^{\sigma(1)})$.
- $\nu' = (\alpha(q-1)^{\sigma(q-1)}, \ldots, \alpha(1)^{\sigma(1)})$.
- $\omega' = (\alpha(\overline{r})^{-\sigma(\overline{r})}, \ldots, \alpha(\overline{1})^{-\sigma(\overline{1})})$

and $U$ is a vector space of dimension $\alpha(\tau)$.

**Proof.** This is an exercise. \qed

Note that because

$$S^\mu U \otimes S^\nu U = \bigoplus_{\tau} (S^\tau U)^{c_{\mu,\nu}}$$
we get that
\[
\dim(S^\mu U \otimes S^\nu U \otimes (S^\omega U)^*_{GL(U)}) = \dim \bigoplus_{\tau} \left((S^\tau U)^{c_{\mu,\nu}} \otimes (S^\sigma U)^*_{GL(U)}\right) = \\
= \dim((S^\omega U)^{c_{\mu,\nu}} \otimes (S^\sigma U)^*_{GL(U)}) = c_{\mu,\nu}^\omega.
\]
So in particular we have
\[
c_{\mu,\nu}^\omega = \dim SI(T_{p,q}^r, \alpha)_\sigma.
\]

**Definition 3.** If \( \lambda \) is a partition, then \( j(\lambda) \) denotes the number of jumps in \( \lambda \).
So if \( \lambda = (a_1^{b_1}, a_2^{b_2}, \ldots, a_r^{b_r}) \) with \( a_r > a_{r-1} > \cdots > a_1 \geq 1 \) and \( b_1, b_2, \ldots, b_r \geq 1 \) then \( j(\lambda) = r \).

**Corollary 2.** If
\[
\frac{1}{j(\mu) + 1} + \frac{1}{j(\nu) + 1} + \frac{1}{j(\omega) > 1}
\]
then
\[
c_{\mu,\nu}^\omega \leq 1.
\]

**Proof.** By proposition 3, we can take the quiver \( T_{p,q}^r \) with \( p-1 = j(\mu), q-1 = j(\nu) \) and \( r = j(\omega) \) and we can find a dimension vector \( \alpha \) and a weight \( \sigma \) such that
\[
\dim SI(Q, \alpha)_\sigma = c_{\mu,\nu}^\omega.
\]
Suppose that
\[
\frac{1}{j(\mu) + 1} + \frac{1}{j(\nu) + 1} + \frac{1}{j(\omega)} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1
\]
then the quiver \( T_{p,q}^r \) is a Dynkin diagram (if we forget the orientation) of type \( A_n, D_n \) or \( E_6, E_7, E_8 \) (this can be directly checked by hand, although the condition \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 \) is not a coincidence because this quantity plays a role when checking whether the Cartan matrix is positive definite). By Gabriel’s Theorem, \( T_{p,q}^r \) is of finite type, so \( GL(\alpha) \) has only finitely many orbits in \( \text{Rep}(Q, \alpha) \). In particular, \( GL(\alpha) \) has a dense orbit in \( \text{Rep}(Q, \alpha) \). Suppose that
\[
f, g \in SI(Q, \alpha)_\sigma
\]
are two semi-invariants of weight \( \sigma \). Then \( f/g \) is a rational invariant which is constant on the dense orbit of \( \text{Rep}(Q, \alpha) \). This shows that \( f/g \) is constant and \( f \) and \( g \) are linearly dependent. This shows that
\[
\dim SI(Q, \alpha)_\sigma = c_{\mu,\nu}^\omega \leq 1.
\]
\[\square\]
5. EXERCISES

Exercise 1. Proof Proposition 3.

Exercise 2. Consider the quiver $Q$ given by

\[
\begin{array}{cccccccc}
1 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & p-1 & \rightarrow & p & \vdots & \rightarrow & p+1 \\
\end{array}
\]

where there are $k$ arrows between $p$ and $p+1$. Let $\alpha$ be a dimension vector with $\alpha(1) \leq \alpha(2) \leq \cdots \leq \alpha(p)$ and $\alpha(p+1) = 1$ and let $\sigma$ be a weight with $\sigma(i) \geq 0$ for $i \leq p$ and $\sigma(p+1) \leq 0$. Prove that

\[\text{SI}(Q, \alpha)_\sigma \cong S^\mu W\]

where $W$ is a vector space of dimension $k$ and

\[\mu' = (\alpha(p)^{\sigma(p)}, \cdots, \alpha(1)^{\sigma(1)}).\]