LECTURE 9: SCHOFIELD SEMI-INVARIANTS

HARM DERKSEN

In this lecture we will introduce semi-invariants which were defined by Schofield. The nice thing about these semi-invariants is that we can associate semi-invariants to representations. Also vanishing of such a semi-invariants has an interesting interpretation in terms Hom and Ext. It will turn out that these particular Schofield semi-invariants span the space of semi-invariants. This will then later have interesting consequences, such as the saturation of Littlewood-Richardson coefficients (i.e., if $c^{M\nu}_{M\lambda,M\mu} > 0$ for some positive integer M, then $c^{\nu}_{\lambda,\mu} > 0$).

1. Definition and basic properties of Schofield semi-invariants

Let Q be a quiver without oriented cycles. Let us now first introduce the Schofield semi-invariants. Remember the Ringel resolution:

$$0 \to \bigoplus_{a \in Q_1} P_{ha} \otimes V(ta) \to \bigoplus_{x \in Q_1} P_x \otimes V(x) \to V \to 0.$$

If we apply $\mathrm{Hom}_Q(\cdot, W)$ we get (as we have seen before) the exact sequence

(1)

$$0 \to \mathrm{Hom}_Q(V,W) \to \bigoplus_{x \in Q_0} \mathrm{Hom}(V(x), W(x)) \xrightarrow{d^V_W} \bigoplus_{a \in Q_1} \mathrm{Hom}(V(ta), W(ha)) \to \mathrm{Ext}_Q(V,W) \to 0.$$

The map d^V_W is defined by

$$\{\phi(x) \mid x \in Q_0\} \mapsto \{W(a)\phi(ta) - \phi(ha)V(a) \mid a \in Q_1\}.$$

Suppose that V has dimension vector α and W is of dimension β. Let us assume that

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha) = 0.$$

Then d^V_W is a square matrix. We now define $c(V,W) := \det d^V_W$. Note that $c \in K[\mathrm{Rep}(Q,\alpha) \times \mathrm{Rep}(Q,\beta)]$ is invariant under $\mathrm{SL}(\alpha)$ and under $\mathrm{SL}(\beta)$. We now define $c^V := c(V,\cdot) \in K[\mathrm{Rep}(Q,\beta)]^{\mathrm{SL}(\beta)} = \mathrm{SI}(Q,\beta)$ and $c_W := c(\cdot, W) \in K[\mathrm{Rep}(Q,\alpha)]^{\mathrm{SL}(\alpha)} = \mathrm{SI}(Q,\alpha)$. The semi-invariants c^V and c_W are called **Schofield semi-invariants**.

Let us compute the weights of c^V and c_W. Let $A = \{A(x) \mid x \in Q_0\} \in \mathrm{GL}(\beta)$. We first view the action of A on $\mathrm{Hom}(V(x), W(x))$. The action of A is given by the blockmatrix

$$\begin{pmatrix} A(x) & & & \\ & A(x) & & \\ & & \ddots & \\ & & & A(x) \end{pmatrix}$$

where there are exactly $\dim(V(x)) = \alpha(x)$ blocks on the diagonal. Similarly, the action of A on $\mathrm{Hom}(V(ta), W(ha))$ is given by a block matrix with $\dim(V(ta)) = \alpha(ta)$ blocks $A(ha)$ on the diagonal. We now now the action of A on $\bigoplus_{x \in Q_0} \mathrm{Hom}(V(x), W(x))$ and on $\bigoplus_{a \in Q_1} \mathrm{Hom}(V(ta), W(ha))$, so we now the action of A on the matrix d^V_W and on the determinant $c^V(W) := c(V, W) = \det(d^V_W)$. Also note that since $\det(d^V_W)$ is a function, the action of A on $\det(d^V_W)$ is defined by $A \cdot \det(d^V_W) = \det(A^{-1} \cdot d^V_W)$. Taking this into account, we get

$$A \cdot c^V(W) = c^V(A^{-1} \cdot W) = \frac{\prod_{x \in Q_0} \det(A(x))^{\alpha(x)}}{\prod_{a \in Q_1} \det(A(ha))^{\alpha(ta)}} c^V(W)$$

This shows that c^V has weight σ where

$$\sigma(x) = \alpha(x) - \sum_{a \in Q_1, ha = x} \alpha(ta) = \langle \alpha, \epsilon_x \rangle,$$

where ϵ_x is the dimension vector given by $\epsilon_x(x) = 1$ and $\epsilon_x(y) = 0$ if $y \neq x$. Remember that for any dimension vector γ we defined $\sigma(\gamma) = \sum_{x \in Q_0} \sigma(x)\gamma(x)$. In this way we could view σ as a function on dimension vectors. So weights and dimension vectors can be viewed as being dual to each other. Now we have $\sigma(\epsilon_x) = \sigma(x) = \langle \alpha, \epsilon_x \rangle$ for all $x \in Q_0$. In particular we can write

$$\sigma = \langle \alpha, \cdot \rangle.$$

Corollary 1. *The semi-invariant c^V lies in $\mathrm{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle}$ for every $V \in \mathrm{Rep}(Q, \alpha)$ and the semi-invariant c_W lies in $\mathrm{SI}(Q, \alpha)_{-\langle \cdot, \beta \rangle}$ for every $W \in \mathrm{Rep}(Q, \beta)$.*

Proof. The first statement follows from the discussion above and the second statement is a similar computation. □

Suppose that $\sigma = \langle \alpha, \cdot \rangle$ where α is a dimension vector and σ is a weight. It may be useful to compute σ if α is given or to compute α if σ is given. Assume that Q is a quiver without oriented cycles, and define $b_{x,y}$ to be the number of arrows between x and y, i.e.,

$$b_{x,y} = \#\{a \in Q_1 \mid ta = x, ha = y\}.$$

Define $p_{x,y}$ to be the number of paths between x and y.

Lemma 1. *Suppose that $\sigma = \langle \alpha, \cdot \rangle$. Then we have*

$$\sigma(x) = \alpha(x) - \sum_{y \in Q_0 \setminus \{x\}} b_{y,x} \alpha(y).$$

and

$$\alpha(x) = \sum_{y \in Q_0} p_{y,x} \sigma(y).$$

Proof. This is an exercise. □

Remark 1. Suppose that V is of dimension α and W is of dimension β such that $\langle \alpha, \beta \rangle = 0$. Then from (1) follows that

$$c^V(W) = 0 \Leftrightarrow \mathrm{Hom}_Q(V, W) \neq 0 \Leftrightarrow \mathrm{Ext}_Q(V, W) \neq 0.$$

Remark 2. One should also note that if we view V as a representation of dimension α rather than an element of $\mathrm{Rep}(Q, \alpha)$, (i.e., we don't have a fixed basis for $V(x)$ for every $x \in Q_0$), then the semi-invariants c^V are only determined up to a constant.

Example 1. Let Q be the quiver

$$1 \longrightarrow 2.$$

A representation $W \in \mathrm{Rep}(Q, (n, n))$ looks like

$$K^n \xrightarrow{\quad A \quad} K^n.$$

Let V be the representation

$$K \longrightarrow 0$$

Then c^V is the determinant of the map

$$\mathrm{Hom}(K, K^n) \to \mathrm{Hom}(K, K^n)$$

given by composition with A. So $c^V = \det(A)$. Indeed, it is also easy to see that $\mathrm{Hom}(V, W) \neq 0$ if and only if $\det(A) = 0$. Note that c^V is a semi-invariant of weight $\langle (1, 0), \cdot \rangle = (1, -1)$.

Example 2. Let Q be the quiver

$$\circ \longrightarrow \circ \longleftarrow \circ.$$

and let β be the dimension vector $(4, 9, 5)$. A general representation of $\mathrm{Rep}(Q, \beta)$ is of the form

$$K^4 \xrightarrow{\quad A \quad} K^9 \xrightarrow{\quad B \quad} K^5$$

Let V be the representation

$$K \xrightarrow{\quad 1 \quad} K \xleftarrow{\quad 1 \quad} K.$$

Now d^V_W is the map

$$\mathrm{Hom}(K, K^4) \oplus \mathrm{Hom}(K, K^9) \oplus \mathrm{Hom}(K, K^5) \to \mathrm{Hom}(K, K^9) \oplus \mathrm{Hom}(K, K^9)$$

which is given by the matrix

$$d^V_W = \begin{pmatrix} A & -\mathrm{id} & 0 \\ 0 & -\mathrm{id} & B \end{pmatrix}$$

By wiping in this matrix, it is easy to see that $c^V = \det(d^V_W) = \pm\det(A \mid B)$. The weight of the semiinvariant is $\langle (1,1,1), \cdot\rangle = (1,-1,1)$.

Example 3. Let Q be the quiver

$$\circ \rightrightarrows \circ$$

and let β be the dimension vector (n,n). A representation in $\mathrm{Rep}(Q,\beta)$ is

$$K^n \underset{B}{\overset{A}{\rightrightarrows}} K^n.$$

Suppose that V is the representation

$$K \underset{\lambda}{\overset{1}{\rightrightarrows}} K.$$

Then $d^V_W : \mathrm{Hom}(K, K^n) \oplus \mathrm{Hom}(K, K^n) \to \mathrm{Hom}(K, K^n) \oplus \mathrm{Hom}(K, K^n)$ is given by the matrix

$$\begin{pmatrix} A & -\mathrm{id} \\ B & -\lambda\,\mathrm{id} \end{pmatrix}.$$

Again by wiping it is clear that $c^V = \det(d^V_W) = \pm\det(B - \lambda A)$. The weight of c^V is $\langle (1,1), \cdot\rangle = (1,-1)$.

2. Schofield-semiinvariants span the ring of semiinvariants

The Schofield semi-invariants also behave nicely with respect to exact sequences as the following proposition shows.

Lemma 2. *Suppose that*

$$0 \to V' \to V \to V'' \to 0$$

and

$$0 \to W' \to W \to W'' \to 0$$

are two exact sequences. Let $\alpha, \alpha', \alpha'', \beta, \beta', \beta''$ be the dimensions of V, V', V'', W, W', W'' respectively. Assume that $\langle \alpha, \beta\rangle = 0$.

(a). *Suppose that $\langle \alpha', \beta\rangle > 0$, then $c^V(W) = 0$.*
(b). *Suppose that $\langle \alpha', \beta\rangle = 0$, then $c^V(W) = c^{V'}(W)c^{V''}(W)$.*
(c). *Suppose that $\langle \alpha, \beta'\rangle < 0$ then $c_W(V) = c^V(W) = 0$.*
(d). *Suppose that $\langle \alpha, \beta'\rangle = 0$ then $c_W(V) = c_W(V')c_W(V'')$.*

Proof. Consider the following diagram with exact columns.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigoplus_{x\in Q_0} \operatorname{Hom}(V''(x), W(x)) & \xrightarrow{d_W^{V''}} & \bigoplus_{a\in Q_1} \operatorname{Hom}(V''(ta), W(ha)) \\
\downarrow & & \downarrow \\
\bigoplus_{x\in Q_0} \operatorname{Hom}(V(x), W(x)) & \xrightarrow{d_W^{V}} & \bigoplus_{a\in Q_1} \operatorname{Hom}(V(ta), W(ha)) \\
\downarrow & & \downarrow \\
\bigoplus_{x\in Q_0} \operatorname{Hom}(V'(x), W(x)) & \xrightarrow{d_W^{V'}} & \bigoplus_{a\in Q_1} \operatorname{Hom}(V'(ta), W(ha)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

(a) If $\langle \alpha', \beta \rangle > 0$ then $d_W^{V'}$ is not surjective, hence d_W^V is not surjective. (b) If $\langle \alpha', \beta \rangle = 0$ then also $\langle \alpha'', \beta \rangle = 0$ and $d_W^{V'}$, d_W^V and $d_W^{V''}$ are all square matrices. Now d_W^V is equal to the blockmatrix

$$\begin{pmatrix} d_W^{V'} & * \\ 0 & d^{V''}{}_W \end{pmatrix}$$

where $*$ stands for some nonzero block. Clearly now $\det(d_W^V) = \det(d_W^{V'}) \det(d_W^{V''})$. (c) and (d) are proven similarly to (a) and (b). □

Theorem 1. *Suppose that Q is a quiver without oriented cycles, β is a dimension vector and σ is a weight. Then $\mathrm{SI}(Q, \beta)_\sigma$ is spanned by c^V's where $V \in \mathrm{Rep}(Q, \alpha)$ and $\sigma = \langle \alpha, \cdot \rangle$.*

Proof. Let us fix Q, β and a weight σ. We proceed in three steps. In the first step, we reduce the theorem to the case that Q is a quiver with exactely one source x_- and one sink x_+ and $\sigma(x_-) = 1$, $\sigma(x_+) = -1$ and σ is zero on all other vertices. In the second step we reduce to the case that there are no vertices x with $\sigma(x) = 0$. The only case left is the quiver Θ_m with weight τ. In Step 3 we will prove the Theorem in this case.

Step 1.

We may assume that $Q_0 = \{x_1, \dots, x_r\}$ such that there are no arrows from x_j to

x_i with $j > i$. Construct a quiver $Q(\sigma)$ as follows.

$$\begin{aligned} Q(\sigma)_0 &= Q_0 \cup x_- \cup x_+ \\ Q(\sigma)_1 &= Q_1 \cup Q_- \cup Q_+ \end{aligned}$$

where Q_- consists of the set of arrows from x_- to x_i, with $\sigma(x_i)$ arrows going to the vertex x_i for which $\sigma(x_i) > 0$ and no arrows going to other vertices. The set Q_+ consists of the set of arrows from x_i to x_+, with $-\sigma(x_i)$ arrows going from the vertex x_i for which $\sigma(x_i) < 0$ and no arrows going from other vertices to x_+.

Example 4. Let Q be the quiver

$$\begin{array}{ccc} x_1 & & \\ & \searrow & \\ & & x_3 \ . \\ & \nearrow & \\ x_2 & & \end{array}$$

Let $\sigma = (1, 1, -2)$. Then the quiver $Q(\sigma)$ is

$$\begin{array}{ccccccc} & & x_1 & & & & \\ & \nearrow & & \searrow & & & \\ x_- & & & & x_3 & \rightrightarrows & x_+ \ . \\ & \searrow & & \nearrow & & & \\ & & x_2 & & & & \end{array}$$

We will write $\overline{Q} = Q(\sigma)$. Define the weight $\overline{\sigma}$ of $\overline{Q}$ by $\overline{\sigma}(x_-) = 1$, $\overline{\sigma}(x_i) = 0$, $\overline{\sigma}(x_+) = -1$. The dimension vector $\overline{\beta} = \beta(\sigma)$ is defined by $\overline{\beta}(x_i) = \beta(x_i)$, $\overline{\beta}(x_-) = \sum_{\{i|\sigma(x_i)>0\}} \sigma(x_i)\beta(x_i)$, $\overline{\beta}(x_+) = \sum_{\{i|\sigma(x_i)<0\}} -\sigma(x_i)\beta(x_i)$. Suppose that $W \in \operatorname{Rep}(\overline{Q}, \overline{\beta})$. The matrices of all maps $W(a)$ with $a \in Q_-$ form a square matrix. Let $D^-(W)$ be the determinant of this block matrix. Let $D^+(W)$ be the determinant of all $W(a)$ with $a \in Q_+$. Then the correspondance $c \to D^- c D^+$ gives the isomorphism of weight spaces $\operatorname{SI}(Q, \beta)_\sigma \to \operatorname{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$.

Let $\overline{\alpha}$ be the dimension vector of $\overline{Q}$ such that $\overline{\sigma} = \langle \overline{\alpha}, \cdot \rangle$. Let $\overline{V}$ be a representation of $\overline{Q}$ with dimension vector $\overline{\alpha}$ and let $c^{\overline{V}}$ be the corresponding non-zero semi-invariant on $\operatorname{SI}(\overline{Q}, \overline{\beta})$.

Proposition 1. *The factor c in the decomposition $c^{\overline{V}} = D^- c D^+$ is of the form c^V for some $V \in \operatorname{Rep}(Q, \alpha)$.*

Proof of proposition. Notice that the weight of D^- is equal to $\langle \gamma_-, \cdot \rangle$ where

$$\gamma_-(x_-) = 1, \quad \gamma_-(x_j) = \gamma_-(x_+) = 0.$$

Similarly, by Lemma 1, the weight of D^+ equals to $\langle \gamma_+, \cdot \rangle$ where

$$\gamma_+(x_-) = 0, \quad \gamma_+(x_j) = -\sum_{\substack{i \leq j \\ \sigma(x_i) < 0}} p_{i,j}\sigma(x_i),$$

$$\gamma_+(x_+) = -1 + \sum_{\substack{j \\ \sigma(x_j)<0}} \sum_{\substack{i\leq j \\ \sigma(x_i)<0}} p_{i,j}\sigma(x_i)\sigma(x_j)$$

It is easy to see that $\langle \gamma_-, \overline{\beta}\rangle = \langle \gamma_+, \overline{\beta}\rangle = 0$.

Let $\overline{V} \in \mathrm{Rep}(\overline{Q}, \overline{\alpha})$. Then $\overline{V}$ has an obvious submodule $\overline{V}_1 = \overline{V}\mid_{\overline{Q}_0\setminus\{x_-\}}$. We have an exact sequence

$$0 \to \overline{V}_1 \to \overline{V} \to \overline{V}_2 \to 0$$

with the dimension of $\overline{V}_2$ equal to γ_-.

Let M be the module defined by the exact sequence

$$0 \to P_{x_+} \xrightarrow{i} \bigoplus_{b, hb=x_+} P_{tb} \to M \to 0,$$

where the morphism i from P_{x_+} to a copy P_{tb} maps the trivial path $e(x_+)$ to the path b. The dimension vector of M is γ_+, and c^M is the determinant D^+. Consider the map

$$\sum_{\substack{b \\ hb=x_+}} \overline{V}_1(b) : \bigoplus_{b,hb=x_+} \overline{V}_1(tb) \to \overline{V}_1(x_+)$$

The dimension of the kernel is at least 1. Let $(s_b)_{b,hb=x_+}$ with $s_b \in \overline{V}_1(tb)$ be a non-trivial element in the kernel. We can define now a map $\bigoplus_{b,hb=x_+} P_{tb} \to \overline{V}_1$ by sending the generator $e(tb) \in P_{tb}(tb)$ to s_b for all b. Because $(s_b)_{b,hb=x_+}$ lies in the kernel, this actually defines a morphism $M \to \overline{V}_1$. Let $\overline{V}_3$ be the image of this morphism.

Now $\overline{V}_3$ is a submodule of $\overline{V}_1$ and $c^{\overline{V}_1} \neq 0$. By Lemma 2 a) we have $\langle \underline{d}(\overline{V}_3), \overline{\beta}\rangle \geq 0$. We also have $c^M = D^+ \neq 0$. If we apply Lemma 2 a) to the kernel N of $M \to \overline{V}_3$, then we get $\langle \underline{d}(N), \overline{\beta}\rangle = \langle \gamma_+, -\underline{d}(\overline{V}_3)\rangle = -\langle \underline{d}(\overline{V}_3), \overline{\beta}\rangle \geq 0$. We conclude $\langle \underline{d}(\overline{V}_3), \overline{\beta}\rangle = 0$. By Lemma 2 b) $c^{\overline{V}_3}$ divides the semi-invariant $c^M = D^+$. Because D^+ is an irreducible semi-invariant we must have $c^{\overline{V}_3} = D^+$, $\gamma_+ = \dim \overline{V}_3$ and $\overline{V}_3$ is isomorphic to M.

We have an exact sequence

$$0 \to \overline{V}_3 \to \overline{V}_1 \to \overline{V}_4 \to 0.$$

Now it is clear that by multiplicative property that $c^{\overline{V}} = c^{\overline{V}_2}c^{\overline{V}_4}c^{\overline{V}_3}$ with the first factor being proportional to D^- and the last one to D^+. Let us also define a submodule $\overline{V}_5 = \overline{V}_4\mid_{\{x_+\}}$, so we have an exact sequence

$$0 \to \overline{V}_5 \to \overline{V}_4 \to \overline{V}_6 \to 0.$$

Note that $\overline{V}_6$ has support within Q. The restriction of $\overline{V}_6$ to Q will be denoted by V. We will prove that the restriction of $c^{\hat{V}}$ to $\mathrm{Rep}(Q, \beta)$ is c^V.

Extend $W \in \text{Rep}(Q,\beta)$ to the module $\overline{W}$ of dimension $\overline{\beta}$ by putting $\overline{W}(x_-) = \bigoplus_{a,ta=x_-} W(ha)$, $\overline{W}(x_+) = \bigoplus_{b,hb=x_+} W(tb)$ with the maps $\overline{W}(a)$ and $\overline{W}(b)$ being the components of the identity map. Define the canonical submodule $\overline{W}_1 = \overline{W}\mid_{\{x_+\}}$. We have an exact sequence

$$0 \to \overline{W}_1 \to \overline{W} \to \overline{W}_2 \to 0.$$

Define the submodule $\overline{W}_3 = \overline{W}_2 \mid_{\hat{Q}\setminus\{x_-\}}$ of $\overline{W}_2$. Now we have an exact sequence

$$0 \to \overline{W}_3 \to \overline{W}_2 \to \overline{W}_4 \to 0.$$

The representation $\overline{W}_3$ has support within Q and its restriction to Q is just W.

We now have

$$c^{\overline{V}}(\overline{W}) = c^{\overline{V}_4}(\overline{W}) = c^{\overline{V}_4}(\overline{W}_1)c^{\overline{V}_4}(\overline{W}_3)c^{\overline{V}_4}(\overline{W}_4) = c^{\overline{V}_4}(\overline{W}_3)$$

because $c^{\overline{V}_4}(\overline{W}_1)$ and $c^{\overline{V}_4}(\overline{W}_4)$ are constant. Moreover,

$$c^{\overline{V}_4}(\overline{W}_3) = c^{\overline{V}_5}(\overline{W}_3)c^{\overline{V}_6}(\overline{W}_3) = c^{\overline{V}_6}(\overline{W}_3) = c^{V}(W)$$

because $c^{\overline{V}_5}(\overline{W}_4)$ is constant. This concludes the proof of the proposition. □

Step 2.
Let Q, β, σ be as above. Let $x \in Q_0$ be a vertex such that $\sigma(x) = 0$. Let $a_1,\dots,a_s$ be the arrows in Q_1 with $ha_k = x$ ($k = 1,\dots,s$) and let $b_1,\dots,b_t$ be the arrows in Q_1 with $tb_l = x$ ($l = 1,\dots,t$). Let $\overline{Q}$ be the quiver such that $\overline{Q}_0 = Q_0 \setminus \{x\}$ and $\overline{Q}_1 = (Q_1 \setminus \{a_1,\dots,a_s,b_1,\dots,b_t\}) \cup \{ba_{k,l}\}_{1\leq k\leq s, 1\leq l\leq t}$, where $t(ba_{k,l}) = ta_k, h(ba_{k,l}) = hb_l$. Let $\overline{\beta}$, $\overline{\sigma}$ be the restrictions of β,σ to $Q_0 \setminus \{x\}$.

The Fundamental Theorem of Invariant Theory (see C. DeConcini, C. Procesi, *Characteristic free approach to invariant theory*, Adv. Math. **21** (1976), 330–354, for a characteristic free version) says that every semi-invariant from $\text{SI}(Q,\beta)_\sigma$ can be obtained from the semi-invariants from $\text{SI}(\overline{Q},\overline{\beta})_{\overline{\sigma}}$ by substituting for the arrows of type $ba_{k,l}$ the actual compositions $b_l a_k$. Assuming Theorem 1 for $\text{SI}(\overline{Q},\overline{\beta})_{\overline{\sigma}}$ to be true, we need to show that every semi-invariant $c^{\overline{V}}$ from $\text{SI}(\overline{Q},\overline{\beta})_{\overline{\sigma}}$ pulls back to a semi-invariant of type c^V. For given representation $\overline{V}$ of $\overline{Q}$ of dimension $\overline{\alpha}$ we define the representation $V = \text{ind}\,\overline{V}$ as follows. We notice that the condition $\sigma(x) = 0$ means that we expect $\dim V(x) = \sum_{k=1}^s \dim V(ta_k)$.

This means we put

$$V(y) = \begin{cases} \overline{V}(y) & \text{if } y \neq x; \\ \bigoplus_{k=1}^s \overline{V}(ta_k) & \text{if } y = x. \end{cases}$$

We define the linear maps $V(a)$ as follows

$$V(a) = \begin{cases} \overline{V}(a) & \text{if } a \neq a_k, b_l; \\ i(a_k) & \text{if } a = a_k; \\ \sum_{k=1}^s \overline{V}(ba_{k,l}) & \text{if } b = b_l \end{cases}$$

where $i(a_k) : V(ta_k) \to \bigoplus_{k=1}^{s} V(ta_k)$ is the injection on the k-th summand.

Then it is easy to check directly from the definition of semi-invariants c^V that if the representation $\overline{W} = \text{res } W$ of dimension $\overline{\beta}$ is a restriction of a representation W of Q of dimension β then $c^{\overline{V}}(\overline{W}) = c^V(W)$.

Notice that the functor ind $\overline{V}$ is the left adjoint of the obvious restriction functor res : $\text{Rep}(Q) \to \text{Rep}(\overline{Q})$, i.e., we have the natural isomorphisms

$$\text{Hom}_Q(\text{ind } \overline{V}, W) = \text{Hom}_{\overline{Q}}(\overline{V}, \text{res } W)$$

which explains why $c^{\overline{V}}(\overline{W})$ and $c^V(W)$ vanish simultaneously.

Step 3.

It remains to deal directly with the weight space $\text{SI}(\Theta_m, \theta(n))_\tau$. Writing the representation W of dimension $\theta(n)$ as an m-tuple of linear maps

$$W(a_1), \dots, W(a_m) : W_- \to W_+$$

we can introduce the additional action of the group $\text{GL}(m)$ acting on this space by taking linear combinations of the linear maps $W(a_1), \dots, W(a_m)$. Using Cauchy formula (in its characteristic free version, say from K. Akin, D. A. Buchsbaum, J. Weyman, *Schur functors and Schur complexes*, Adv. Math. **44** (1982), 207–278.) we see that the space $\text{SI}(\Theta_m, \theta(n))_\tau$ of semi-invariants can be identified with $\bigwedge^n W_- \otimes \bigwedge^n W_+^* \otimes D_n(K^m)$. Here D_n denotes the n-th divided power. Since the divided power $D_n(K^m)$ is generated as an $\text{GL}(m)$-module by its highest weight vector (which corresponds to the semi-invariant $\det W(a_1)$) and the set of semi-invariants of the form c^V is preserved by the action of $\text{GL}(m)$, it is enough to express $\det W(a_1)$ as the semi-invariant of the form c^V. Notice that $\tau = \langle \alpha, \cdot \rangle$ for the dimension vector $\alpha = (1, m-1)$. Taking the module V to be the m-tuple of linear maps $V(a_1), \dots, V(a_m) : K \to K^{m-1}$ where $V(a_1) = 0$ and $V(a_i)$ is the embedding sending 1 to the $i-1$'st basis vector, for $i = 2, \dots, m$, we check directly that $c^V = \det\ W(a_1)$. This concludes the proof of Theorem 1. □

Corollary 2. (Reciprocity Property). *Let α, β be two dimension vectors for the quiver Q. Assume that $\langle \alpha, \beta \rangle = 0$. Then*

$$\dim_K \text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} = \dim_K \text{SI}(Q, \alpha)_{-\langle \cdot, \beta \rangle}.$$

Proof. Let $V_1, \dots, V_s$ be the modules of dimension α such that $c^{V_1}, \dots c^{V_s}$ form a basis of $\text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle}$. These are linearly independent polynomials on $\text{Rep}(Q, \beta)$ so there exist s representations $W_1, \dots, W_s$ in $\text{Rep}(Q, \beta)$ such that $\det(c^{V_i}(W_j))_{1 \le i,j \le s}$ is not zero. But $c^{V_i}(W_j) = c_{W_j}(V_i)$ and this means that the semi-invariants $c_{W_1}, \dots, c_{W_s}$ are linearly independent. This proves that

$$\dim_K \text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} \le \dim_K \text{SI}(Q, \alpha)_{-\langle \cdot, \beta \rangle}.$$

The other inequality is proven in exactly the same way. □

3. EXERCISES

Exercise 1. Proof Lemma 1.

Exercise 2. (a). Suppose that $0 \to P_1' \to P_0' \to V \to 0$ is the Ringel resolution and $0 \to P_1 \to P_0 \to V \to 0$ is the minimal resolution. Show that we have the following diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & P & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1' & \longrightarrow & P_0' & \longrightarrow & V & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(b). Suppose that V is α dimensional and β is a dimension vector with $\langle \alpha, \beta \rangle = 0$. Show that c^V can also be defined in the following way: Take the *minimal* resolution $0 \to P_1 \to P_0 \to V \to 0$ and define $c^V(W)$ as the determinant of

$$\mathrm{Hom}_Q(P_0, W) \to \mathrm{Hom}_Q(P_1, W)$$

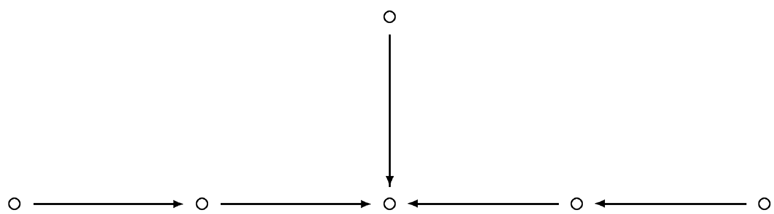
for every $W \in \mathrm{Rep}(Q, \beta)$.

Exercise 3. (a). Let α, β be two dimension vectors for a quiver Q without oriented cycles. Show that $\mathrm{SI}(Q, \beta)$ is generated by c^V's with V indecomposable.

(b). Suppose that V_1 and V_2 are representations of a quiver Q without oriented cycles of dimension α_1 and α_2 respectively and assume that $\langle \alpha_1, \beta \rangle = \langle \alpha_2, \beta \rangle = 0$ for some dimension vector β. Suppose that the semi-invariants $c^{V_1}, c^{V_2} \in \mathrm{Rep}(Q, \beta)$ are nonzero. Let $\phi : V_1 \to V_2$ a morphism of quiver representations let let V_3 be its image. Prove that $c^{V_3} \neq 0$ and c^{V_3} divides c^{V_1} and c^{V_2}.

(c). Show that $\mathrm{SI}(Q,\beta)$ is generated by c^V's satisfying $\mathrm{Hom}_Q(V,V) = K$ (this actually implies (a)). Note: A representation V with $\mathrm{Hom}_Q(V,V) = K$ is sometimes called a **Schur representation** (named after Schur's Lemma). (Hint: If $\mathrm{Hom}_Q(V,V) \neq K$ then we can find a morphism $\phi : V \to V$ which is nonzero, but not an isomorphism).

Exercise 4. Find generators of $\mathrm{SI}(Q,\beta)$ where Q is the quiver



and the dimension vector

$$\beta = \begin{matrix} & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{matrix}.$$

Use c^V where V is indecomposable. Find all roots α with $\langle \alpha, \beta \rangle = 0$. Then determine which of these roots α actually give nonzero irreducible semi-invariants c^V if we take $V \in \mathrm{Rep}(Q,\alpha)$. Then use the minimal resolutions of those V's to compute c^V explicitly.