DEGREE BOUNDS FOR SYZYGIES OF INVARIANTS

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ABSTRACT. Suppose that $G$ is a linearly reductive group. Good degree bounds for generators of invariant rings were given in [2]. Here we study the minimal free resolution of the invariant ring. Recently it was shown that if $G$ is a finite linearly reductive group, then the ring of invariants is generated in degree $\leq |G|$ (see [5, 6, 3]). This extends the classical result of Noether who proved the bound in characteristic 0 (see [9]). We will prove that for a finite linearly reductive group $G$, the ideal of relations of a minimal set of generators of the invariant ring is generated in degree $\leq 2|G|$.

1. INTRODUCTION

Let us fix a linearly reductive group $G$ over a field $K$. If $V$ is a representation of $G$, then $G$ acts on the coordinate ring $K[V]$. The ring of invariant functions is denoted by $K[V]^G$. We define the constant $\beta_G(V)$ as the smallest positive integer $d$ such that all invariants of degree $\leq d$ generate the invariant ring.

In characteristic 0, E. Noether proved for finite groups $G$ that $\beta_G(V) \leq |G|$ where $|G|$ is the order of the group. A group $G$ over a field $K$ is linearly reductive if and only if the characteristic of the base field does not divide the group order $G$. This situation is often referred to as the non-modular case. Noether's bound was recently extended by Fleischmann (see [5]) to the general non-modular case. Fogarty found another proof of this independently in [6]. A third proof follows from the subspaces conjecture in [1], which was solved by Sidman and the author in [3].

For connected linearly reductive groups, upper bounds for $\beta_G(V)$ which depend polynomially on the weights appearing in the representation were given in [2].

In this paper, we discuss good degree bounds for the higher syzygies of this invariant ring. Suppose that $f_1, f_2, \ldots, f_r$ are minimal generators of the invariant ring $R = K[V]^G$ whose degrees are $d_1 \geq d_2 \geq \cdots \geq d_r$. We define the graded polynomial ring $S = K[x_1, x_2, \ldots, x_r]$ where $\deg(x_i) = d_i$ for all $i$. Now $R$ is an $S$-module via the surjective ring homomorphism $\varphi : S \rightarrow R$ defined by $\varphi(x_i) = f_i$ for all $i$. Let

$$0 \rightarrow F_k \rightarrow F_{k-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow R \rightarrow 0$$

be the minimal free graded resolution of $R$ as an $S$-module. We define $\beta_{S}(V)$ as the smallest integer such that $F_i$ is generated as an $S$-module in degree at most $d$. Note that $F_0 = S$ and the image of $F_1$ in $F_0$ is the syzygy ideal $J$ of $S$ defined by

$$J = \{ h \in k[x_1, \ldots, x_r] \mid h(f_1, f_2, \ldots, f_r) = 0 \}.$$

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The invariant ring $K[V]^G$ is Cohen-Macaulay (see [7]). By standard methods, one can estimate the degrees in the minimal resolution of $K[V]^G$ by cutting the invariant ring down by hypersurfaces. Using an estimate of Knop for the $a$-invariant of the invariant ring, we will prove the following bound:

**Theorem 1.** We have

$$\beta_G^a(V) \leq d_1 + d_2 + \cdots + d_{s+i} - s \leq (s + i)\beta_G(V) - s$$

where $s$ is the Krull dimension of $K[V]^G$.

For connected linearly reductive groups $G$, this theorem gives polynomial bounds for $\beta_G^a(V)$ as well. In particular $J$ is generated in degree at most $\beta_G^a(V) \leq (s + 1)\beta_G(V) - s$. For finite groups in the nonmodular case, we get $\beta_G^a(V) \leq (n + i)|G| - n$, and $J$ is generated in degree at most $\beta_G^a(V) \leq (n + 1)|G| - n$. This last inequality can be improved.

For a finite group in the nonmodular case, we define $\tau_G(V)$ as the smallest positive integer $d$ such that every polynomial of degree $d$ appears in the ideal $I$ generated by all homogeneous invariants of positive degree. Recall that the Castelnuovo-Mumford regularity of a graded $R$-module $M$ is the smallest integer $d$ for which a free resolution

$$0 \to H_i \to H_{i-1} \to \cdots \to H_0 \to M \to 0$$

exists, where $H_i$ is generated in degree $\leq d + i$ for all $i$. It is well-known that the Castelnuovo-Mumford regularity of a finite length graded $R$-module $M$ is exactly the maximum degree appearing in $M$ (see [4, Exercise 20.15] or Theorem 6). In particular, the Castelnuovo-Mumford regularity of $R/I$ is $\tau_G(V) - 1$. From [4, Corollary 20.19 a)] and the exact sequence $0 \to I \to R \to R/I \to 0$ follows that the regularity of $I$ is $\leq \tau_G(V)$ (in fact from the long exact sequence even follows equality). From for example Fogarty’s proof of the Noether bound in the nonmodular case (see [6]), follows that $\tau_G(V) \leq |G|$. The following theorem gives an improved bound for the regularity of the syzygy ideal.

**Theorem 2.** Suppose that $G$ is a finite group in the nonmodular case. Suppose that \{f_1, f_2, \ldots, f_r\} is a minimal set of homogeneous generators of the invariant ring $K[V]^G$ and let $J \subseteq K[x_1, x_2, \ldots, x_r]$ be the syzygy ideal. Then $J$ is generated in degree at most

$$2\tau_G(V) \leq 2|G|.$$
2. Examples

Example 4. Let $S_n$ be the symmetric group acting on $V = K^n$ by permutation of the coordinates and let $A_n \subset S_n$ be the alternating group. The coordinate ring $K[V]$ can be identified with the polynomial ring $K[y_1, \ldots, y_n]$. The invariant ring of $S_n$ is given by

$$K[y_1, y_2, \ldots, y_n]^{S_n} = K[e_1, e_2, \ldots, e_n]$$

where $e_j$ is the elementary symmetric polynomial of degree $j$ defined by

$$e_j = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} y_{i_1} y_{i_2} \cdots y_{i_j}$$

for all $j$.

It is also well known that the ring of invariants of $A_n$ is given by

$$K[y_1, \ldots, y_n]^{A_n} = K[e_1, e_2, \ldots, e_n, \Delta]$$

where $\Delta$ is an $A_n$-invariant of degree $n(n - 1)/2$ defined by

$$\Delta = \prod_{1 \leq i < j \leq n} (y_i - y_j).$$

Let us define again a surjective ring homomorphism

$$K[x_1, x_2, \ldots, x_{n+1}] \to K[y_1, \ldots, y_n]^{A_n}$$

where $x_i$ maps to $e_i$ for $i \leq n$ and $x_{n+1}$ maps to $\Delta$. The kernel of the homomorphism is the syzygy ideal $J$.

Theorem 1 says that $J$ is generated in degree

$$\leq n(n - 1)/2 + n + (n - 1) + \cdots + 1 - n = n(n - 1).$$

Note that every polynomial of degree $> n(n - 1)/2$ lies in the ideal $(e_1, e_2, \ldots, e_n)$ of $K[y_1, \ldots, y_n]$. This follows from the fact that $e_1, e_2, \ldots, e_n$ is a regular sequence. In fact, the Hilbert series of $K[y_1, \ldots, y_n]/(e_1, e_2, \ldots, e_n)$ is equal to

$$(1 + t)(1 + t + t^2) \cdots (1 + t + \cdots + t^{n-1}) = 1 + (n-1)t + \cdots + t^{n(n-1)/2}$$

In particular $K[y_1, \ldots, y_n]/(e_1, \ldots, e_n)$ is 1-dimensional in degree $n(n - 1)/2$. This one dimensional space is generated by the invariant $\Delta$. This shows that every polynomial of degree $\geq n(n - 1)/2$ lies in the ideal $(e_1, e_2, \ldots, e_n, \Delta)$, so $\tau_{A_n}(V) = n(n - 1)/2$. By Theorem 2 we get that $J$ is generated in degree

$$\leq 2 \cdot n(n - 1)/2 = n(n - 1).$$

Both theorems are sharp in this example. We have that $\Delta^2$ is $S_n$-invariant and therefore $\Delta^2$ is a polynomial in $e_1, e_2, \ldots, e_n$. This gives a relation of degree $n(n - 1)$ and it is known that this relation generates the ideal $J$. 
Example 5. Let $G$ be the cyclic group of order $m$, generated by $\sigma$. Let $\sigma$ act on $V = K^n$ by scalar multiplication with a primitive $m$-th root of unity $\zeta$. This defines a group action of $G$ on $V$. Again $K[V] = K[y_1, \ldots, y_n]$ where $y_i$ is the $i$-th coordinate function. The invariant ring $K[V]^G$ is generated by the set $\mathcal{M}$ of all monomials in $y_1, y_2, \ldots, y_n$ of degree $m$. To every monomial $M \in \mathcal{M}$ we attach a formal variable $x_M$. We consider the surjective ring homomorphism

$$K[\{x_M\}_{M \in \mathcal{M}}] \twoheadrightarrow K[V]^G$$

which maps $x_M$ to $M$ for every monomial $M \in \mathcal{M}$. The kernel of this homomorphism is again the syzygy ideal $J$.

By Theorem 1, $J$ is generated in degree $\leq (n + 1)m - n$. Since every monomial of degree $m$ lies the ideal $J$ generated by all homogeneous invariants of positive degree, we have $\tau_G(V) = m$. By Theorem 2, $J$ is generated in degree $\leq 2m$ (which means that $J$ is generated by polynomials which are quadratic in the variables $\{x_M\}_{M \in \mathcal{M}}$). Indeed, $J$ is generated by relations of the form

$$x_{y_iM}x_{y_jN} - x_{y_jM}x_{y_iN}$$

where $M, N$ are monomials of degree $m - 1$. Now Theorem 2 is still sharp, but Theorem 1 is not.

3. A general degree bound for syzygies

Suppose that $S = K[x_1, x_2, \ldots, x_r]$ is the graded polynomial ring where $\deg(x_i) = d_i$ is a positive integer for all $i$. We will assume that $d_1 \geq d_2 \geq \cdots \geq d_r$. Suppose that $M$ is a finitely generated graded Cohen-Macaulay $S$-module. The minimal resolution of $M$ is

$$0 \to F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

where

$$F_i \cong \text{Tor}_i^S(M, K) \otimes_K S.$$ 

Here $\text{Tor}_i^S(M, K)$ is a finite dimensional graded vector space, and this makes $\text{Tor}_i^S(M, K) \otimes_K S$ into a graded module.

We will use the following conventions: If $M$ is a finite dimensional graded vector space, then $\deg(M)$ is the maximal degree appearing in $M$ if $M$ is nonzero, and $\deg(M) = -\infty$ if $M$ is zero. For a finitely generated module $M$, $a(M)$ is the degree of the Hilbert series $H(M, t)$, seen as a rational function (the so-called $a$-invariant of $M$).

Theorem 6. We have the inequality

$$\deg(\text{Tor}_i^S(M, K)) \leq d_1 + d_2 + \cdots + d_{s+i} + a(M)$$

where $s$ is the dimension of $M$.

Proof. We prove the theorem by induction on $s = \dim M$. Suppose that $M$ has dimension 0. In this case we prove the inequality by induction of the length $\dim_K M$ of $M$. If $M$ has length 0, then $M$ is the trivial module and the inequality is obvious. Suppose
that $M$ is nonzero. Note that $a := a(M)$ is the maximum degree appearing in $M$. Let $M_a$ be the part of $M$ of degree $a$. Then $M_a$ is a submodule of $M$. We have an exact sequence of $S$-modules

$$0 \to M_a \to M \to M/M_a \to 0$$

Since $\dim_K M/M_a < \dim_K M$ and $a(M/M_a) < a$ we get by induction that

$$\text{deg}(\text{Tor}^S_i(M/M_a, K)) \leq d_1 + \cdots + d_i + a - 1.$$ 

The submodule $M_a$ is isomorphic to the module $K^m[-a]$ which is the module $K^m$ whose degree is shifted by $a$. Since

$$\text{deg}(\text{Tor}^S_i(K, K)) \leq d_1 + d_2 + \cdots + d_i$$

by the Koszul resolution, we have that

$$\text{deg}(\text{Tor}^S_i(M_a, K)) \leq d_1 + d_2 + \cdots + d_i + a.$$ 

From the long exact sequence

$$\cdots \to \text{Tor}^S_i(M_a, K) \to \text{Tor}^S_i(M, K) \to \text{Tor}^S_i(M/M_a, K) \to \cdots$$

follows that

$$\text{deg}(\text{Tor}^S_i(M, K)) \leq d_1 + d_2 + \cdots + d_i + a.$$ 

Now suppose that $s > 0$. Since $M$ is Cohen-Macaulay we can find a homogeneous nonzero divisor $p$ of degree $e > 0$ and $M/pM$ is again Cohen Macaulay. First, note that $H(M/pM, t) = (1 - t^e)H(M, t)$, so $a(M/pM) = a(M) + e$. From the short exact sequence

$$0 \to M[-e] \to M \to M/pM \to 0$$

we obtain a long exact sequence

$$\cdots \to \text{Tor}^S_i(M/pM, K) \to \text{Tor}^S_i(M, K)[-e] \to \text{Tor}^S_i(M, K) \to \cdots$$

Any element of $\text{Tor}^S_i(M, K)[-e]$ of maximal degree must map to 0 in $\text{Tor}^S_i(M, K)$, and therefore it must come from $\text{Tor}^S_{i+1}(M/pM, K)$. This shows that

$$e + \text{deg}(\text{Tor}^S_i(M, K)) = \text{deg}(\text{Tor}^S_i(M, K)[-e]) \leq \text{deg}(\text{Tor}^S_{i+1}(M/pM, K)) \leq$$

$$\leq d_1 + d_2 + \cdots + d_{(s-1)+(i+1)} + a(M/pM) = d_1 + d_2 + \cdots + d_{s+i} + a(M) + e,$$

so finally

$$\text{deg}(\text{Tor}^S_i(M, K)) \leq d_1 + d_2 + \cdots + d_{s+i} + a(M).$$ 

\[ \square \]

Proof of Theorem 1. Let us choose $M = R$ in the previous theorem. Then

$$\beta^G_s(V) = \text{deg}(\text{Tor}^S_i(M, K)) \leq d_1 + d_2 + \cdots + d_{s+i} + a(R)$$

Knop proved that $a(R) \leq -s$ (see [8]) and Theorem 1 follows. \[ \square \]
4. BOUNDS FOR THE SYZYGY IDEAL FOR FINITE GROUPS

Proposition 7. Suppose that \( R = \bigoplus_{d \geq 0} R_d \) is a graded ring with \( R_0 = K \) and that \( \{ f_1, f_2, \ldots, f_r \} \) is a minimal set of homogeneous generators of \( R \). Let \( S = K[x_1, \ldots, x_r] \) be the graded polynomial ring and let \( \varphi : S \to R \) be the surjective ring homomorphism defined by \( x_i \mapsto f_i \) for all \( i \). We have an exact sequence of graded vector spaces

\[
\text{Tor}^S_2(K, K) \to \text{Tor}^S_2(K, K) \to \text{Tor}^S_2(K, K) \to \text{Tor}^S_2(R, K) \to 0
\]

Proof. From Exercise A3.47 (with the role of \( R \) and \( S \) interchanged) in [4], we get a five-term exact sequence

\[
\text{Tor}^R_2(K, K) \to \text{Tor}^S_2(K, K) \to \text{Tor}^S_1(R, K) \to \text{Tor}^S_1(K, K) \to \text{Tor}^R_1(K, K) \to 0.
\]

Let \( n = (x_1, \ldots, x_r) \) be the maximal homogeneous ideal of \( S \) and let \( m = (f_1, \ldots, f_r) \) be the maximal homogeneous ideal of \( R \). Now \( \text{Tor}^S_1(K, K) \) and \( \text{Tor}^S_1(K, K) \) can be identified with \( n/n^2 \) and \( m/m^2 \) respectively. In particular, both \( \text{Tor}^S_1(K, K) \) and \( \text{Tor}^R_1(K, K) \) are \( r \)-dimensional. The proposition follows.

Let us give a more explicit proof. Note that \( \text{Tor}^S_1(R, K) \) is isomorphic to \( J/nJ \). In other words, \( \text{Tor}^S_1(R, K) \) corresponds to the minimal set of generators of \( J \).

If we look at the (possibly infinite) minimal free resolution of \( K \) as an \( R \)-module, we get

\[
\cdots E_2 \to E_1 \to E_0 \to K \to 0
\]

where \( E_0 = R, \ E_1 \cong R[-d_1] \oplus R[-d_2] \oplus \cdots \oplus R[-d_r] \). The kernel \( M \) of \( E_1 \to E_0 \) is the set of all \( (u_1, u_2, \ldots, u_r) \in R[-d_1] \oplus \cdots \oplus R[-d_r] \) with \( u_1f_1 + \cdots + u_rf_r = 0 \). One can identify \( M/mM \) with \( \text{Tor}^R_2(K, K) \).

Similarly we can look at the minimal Koszul resolution of \( K \) as an \( S \)-module:

\[
\cdots D_2 \to D_1 \to D_0 \to K \to 0
\]

where \( D_0 = S, \ D_1 \cong S[-d_1] \oplus S[-d_2] \oplus \cdots \oplus S[-d_r] \). The kernel \( N \) of \( D_1 \to D_0 \) is the set of all \( (v_1, v_2, \ldots, v_r) \in S[-d_1] \oplus \cdots \oplus S[-d_r] \) with \( v_1x_1 + \cdots + v_rx_r = 0 \). Again we can identify \( N/nN \) with \( \text{Tor}^S_2(K, K) \), where \( n \) is the maximal homogeneous ideal of \( S \).

The ring homomorphism \( \varphi : S \to R \) clearly induces maps \( N \to M, \ N/nN \to M/mM \) and therefore it induces a map from \( \text{Tor}^S_2(K, K) \to \text{Tor}^R_2(K, K) \).

Let us finally explain the map \( \psi : \text{Tor}^R_2(K, K) \cong M/mM \to \text{Tor}^S_2(R, K) \cong J/nJ \).

Consider an element \( (u_1, u_2, \ldots, u_r)+mM \in M/mM \). We can find \( v_1 \in S[-d_1], \ldots, v_r \in S[-d_r] \) such that \( \varphi(v_i) = u_i \) for all \( i \). We define

\[
\psi((u_1, \ldots, u_r)+mM) = \sum_{i=1}^r v_ix_i + nJ.
\]

First, note that \( \sum_{i=1}^r v_ix_i \in J \), because \( \varphi(\sum_{i=1}^r v_ix_i) = \sum_{i=1}^r u_if_i = 0 \). The map \( \psi \) is well-defined: If \( \varphi(\tilde{v}_i) = \varphi(v_i) = u_i \) for all \( i \), then \( \tilde{v}_i - v_i \in J \). In particular, the righthandside does not depend on the choice of the \( v_i \)'s. Moreover, suppose that

\[
(\tilde{u}_1, \ldots, \tilde{u}_r) - (u_1, \ldots, u_r) = f_j(u'_1, \ldots, u'_r) \in f_jM
\]
for some $j$ and some $(u'_1, \ldots, u'_r) \in M$. We can find $v'_1, \ldots, v'_r \in S$ such that $\varphi(v'_i) = u'_i$. If we take $\tilde{v}_i = v_i + v'_ix_j$ for all $i$, then $\varphi(\tilde{v}_i) = \tilde{u}_i$ for all $i$ and

$$
\sum_{i=1}^r \tilde{v}_ix_i + nJ = \sum_{i=1}^r v'_ix_i + nJ
$$

because

$$
x_j \sum_{i=1}^r v'_ix_i \in nJ.
$$

Since $m$ is generated by all $f_j$'s, this shows that the map $\psi$ also does not depend on the choice of the representants $u_1, \ldots, u_r$.

The map $\psi$ is surjective. Indeed, if $h \in J$, then we can write $h = v_1x_1 + \cdots + v_rx_r$. We see that

$$
\psi((\varphi(v_1), \ldots, \varphi(v_r)) + mM) = h + nJ.
$$

Notice that

$$
\varphi(v_1)f_1 + \cdots \varphi(v_r)f_r = \varphi(v_1x_1 + \cdots + v_rx_r) = \varphi(h) = h(f_1, f_2, \ldots, f_r) = 0.
$$

We describe the kernel of $\psi$. If $\psi((u_1, u_2, \ldots, u_r) + mM) = 0$, then we can find $v_1, v_2, \ldots, v_r$ such that $\varphi(v_i) = u_i$, and

$$
\sum_{i=1}^r v_ix_i \in nJ.
$$

By modifying $v_1, v_2, \ldots, v_r$ by elements of $J$, we can actually assume that

$$
\sum_{i=1}^r v_ix_i = 0.
$$

Now $(v_1, v_2, \ldots, v_r)$ lies in the module $N$. This shows that the kernel of $\psi$ is contained in the image of $\text{Tor}^S_2(K, K) \to \text{Tor}^S_2(K, K)$. It is also obvious that the composition $\text{Tor}^S_2(K, K) \to \text{Tor}^R_2(K, K) \to \text{Tor}^S_1(R, K)$ is equal to 0.

Proof of Theorem 2. Let us write $T = K[V]$. We consider the $T$-module $U$, defined by

$$
U = \{(w_1, w_2, \ldots, w_r) \in T[-d_1] \oplus \cdots \oplus T[-d_r] \mid \sum_{i=1}^r w_if_i = 0\}.
$$

Since $I = (f_1, f_2, \ldots, f_r)$ is $\tau_G(V)$-regular (in the sense of Mumford and Castelnuovo), we get that $U$ is generated in degree $\leq \tau_G(V) + 1$. Note that $U^G = M$. We have that $(f_1, f_2, \ldots, f_r)U^G = (f_1, f_2, \ldots, f_r)U^G = mM$ since $f_1, \ldots, f_r$ are invariant and $G$ is linearly reductive. We can view $M/mM$ as a submodule of $U/(f_1, \ldots, f_r)U$. It is easy to see that every element of $U$ of degree $\geq 2\tau_G(V) + 1$, must lie in $(f_1, \ldots, f_r)U$ since $U$ is generated in degree $\leq \tau_G(V) + 1$ and every polynomial of degree $\geq \tau_G(V)$ lies in $(f_1, \ldots, f_r)$. This shows that

$$
\deg(\text{Tor}^R_2(K, K)) = \deg(M/mM) \leq \deg(U/(f_1, \ldots, f_r)U) \leq 2\tau_G(V).
$$
By the previous proposition, we also get $\deg(\text{Tor}_1^S(R, K)) \leq 2\tau_G(V)$.

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