ON THE σ-STABLE DECOMPOSITION OF QUIVER REPRESENTATIONS

HARM DERKSEN AND JERZY WEYMAN

1. Introduction

Let Q be a quiver without oriented cycles and let α be a dimension vector. In the paper [2] we proved some results about the set $\Sigma(Q,\alpha)$ of weights occurring in the ring $\mathrm{SI}(Q,\beta)$ of semi-invariants on the space of α-dimensional representations $\mathrm{Rep}(Q,\alpha)$. We showed that this set is given by one linear homogeneous equation and a finite set of homogeneous linear inequalities. Thus it forms a rational polyhedral cone in the space of weights of Q. Still, some of the inequalities given in [2] are redundant. It is an interesting question to determine which inequalities are really needed. In this paper we address this question. We give a description of *all* the faces of arbitrary dimension of the cone $\Sigma(Q,\alpha)$.

An interesting special case is where Q is a triple flag quiver. In that case the cone $\Sigma(Q,\alpha)$ corresponds to all triples of partitions (λ,μ,ν) such that the corresponding Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ is nonzero. Klyachko gave an explicit set of inequalities for this cone. However, some of these inequalities are redundant. A (yet) unpublished result of Knutson, Tao and Woodward gives the set of necessary inequalities for the Klyachko cone. Our results applied to the triple flag quiver give a similar result. Moreover, we find a description for *all* the faces of arbitrary dimension of the Klyachko cone.

Our main tool is a generalization of the canonical decomposition of a dimension vector (see [5]) to arbitrary orthogonal categories. It is based on the notion of stability introduced in this context by King in [4]. For a given weight σ where β, we say that $\alpha = \alpha_1 \dotplus \alpha_2 \dotplus \ldots \dotplus \alpha_s$ is a σ-stable decomposition of α if $\sigma(\alpha_i) = 0$ and α_i is σ-stable for $i = 1,\ldots,s$, and a generic representation of dimension α has a composition series with factors of dimensions $\alpha_1,\ldots,\alpha_s$. The existence and uniqueness of this decomposition follow from general stability results. We use the approach of Rudakov from [7].

The σ-stable decomposition turns out to be related to the faces of the cone $\Sigma(Q,\alpha)$ as follows. Let α be a dimension vector. Then σ is in the relative interior of the face of $\Sigma(Q,\alpha)$ of dimension $n-k$ if and only if exactly k distinct dimension vectors appear in the σ-stable decomposition of β. If $\beta = c_1\cdot\alpha_1 \dotplus c_2\cdot\alpha_2 \dotplus\cdots\dotplus c_k\cdot\alpha_k$ is the σ-stable decomposition with $c_i > 0$ for all i, then the face containing σ is described by the linear equations $\tau(\alpha_i) = 0$, $1 \le i \le k$. All the faces of the cone $\Sigma(Q,\alpha)$ can be obtained in this way.

The first author was supported by NSF, grant DMS 9970165 and the second author was supported by NSF, grant DMS 9700884.

2. Preliminaries

2.1. **Basic notions for quivers.** A quiver Q is a pair $Q=(Q_0,Q_1)$ consisting of the set of vertices Q_0 and the set of arrows Q_1. Each arrow a has its head ha and tail ta, both in Q_0;

$$ta\xrightarrow{a}ha.$$

We fix an algebraically closed field K. A representation V of Q is a family of finite dimensional vector spaces $\{V(x)\mid x\in Q_0\}$ and of linear maps

$$\{V(a):V(ta)\to V(ha)\mid a\in Q_1\}.$$

The dimension vector of a representation V is the function $\underline{d}_V:Q_0\to\mathbb{Z}$ defined by $\underline{d}_V(x):=\dim V(x)$. The dimension vectors lie in the space Γ of integer-valued functions on Q_0. A morphism $\phi:V\to W$ of two representations is the collection of linear maps $\phi(x):V(x)\to W(x)$ such that for each $a\in Q_1$ we have $W(a)\phi(ta)=\phi(ha)V(a)$. We denote the linear space of morphisms from V to W by $\operatorname{Hom}_Q(V,W)$.

From now on we will assume that Q has no oriented cycles, i.e., there is no sequence $a_1,a_2,\ldots,a_n\in Q_1$ of arrows such that $ha_i=ta_{i+1}$ for $i=1,2,\ldots,n-1$ and $ha_n=ta_1$.

The category $\operatorname{Rep}_K(Q)$ of representations of Q is hereditary, i.e., the subobject of a projective object is projective. This means that every representation has projective dimension ≤ 1. The spaces $\operatorname{Hom}_Q(V,W)$ and $\operatorname{Ext}_Q(V,W)$ can be calculated as the kernel and cokernel of the following linear map

$$d^V_W:\bigoplus_{x\in Q_0}\operatorname{Hom}(V(x),W(x))\longrightarrow\bigoplus_{a\in Q_1}\operatorname{Hom}(V(ta),W(ha)) \tag{1}$$

where the map d^V_W restricted to $\operatorname{Hom}(V(x),W(x))$ is given by

$$\{f(x):V(x)\to W(x)\mid x\in Q_0\}\mapsto\{W(a)f(ta)-f(ha)V(a):V(ta)\to W(ha)\mid a\in Q_1\}$$

Let α,β be two elements of Γ. We define the Euler inner product

$$\langle\alpha,\beta\rangle=\sum_{x\in Q_0}\alpha(x)\beta(x)-\sum_{a\in Q_1}\alpha(ta)\beta(ha). \tag{2}$$

It follows from (1) and (2) that

$$\langle\underline{d}_V,\underline{d}_W\rangle=\dim_K\operatorname{Hom}_Q(V,W)-\dim_K\operatorname{Ext}_Q(V,W).$$

2.2. **Semi-invariants for quiver representations.** For a dimension vector β we denote by

$$\operatorname{Rep}(Q,\beta):=\bigoplus_{a\in Q_1}\operatorname{Hom}(K^{\beta(ta)},K^{\beta(ha)})$$

the vector space of representations of Q of dimension vector β. The groups

$$\operatorname{GL}(Q,\beta):=\prod_{x\in Q_0}\operatorname{GL}(\beta(x))$$

and its subgroup

$$\operatorname{SL}(Q,\beta)=\prod_{x\in Q_0}\operatorname{SL}(\beta(x))$$

act on $\operatorname{Rep}(Q,\beta)$ in an obvious way. We are interested in the rings of semi-invariants

$$\operatorname{SI}(Q,\beta)=K[\operatorname{Rep}(Q,\beta)]^{\operatorname{SL}(Q,\beta)}.$$

The ring $\mathrm{SI}(Q,\beta)$ has a weight space decomposition

$$\mathrm{SI}(Q,\beta)=\bigoplus_{\sigma}\mathrm{SI}(Q,\beta)_{\sigma}$$

where σ runs through the characters of $\mathrm{GL}(Q,\beta)$ and

$$\mathrm{SI}(Q,\beta)_{\sigma}=\{f\in K[\mathrm{Rep}(Q,\beta)]\mid g(f)=\sigma(g)f\ \forall g\in\mathrm{GL}(Q,\beta)\}$$

A character or weight of $\mathrm{GL}(\beta)$ has the form

$$\{A(x)\mid x\in Q_0\}\mapsto\prod_{x\in Q_0}(\det A(x))^{\sigma(x)}$$

with $\sigma(x)\in\mathbb{Z}$ for all $x\in Q_0$. If $\alpha\in\Gamma$ is a dimension vector then we define

$$\sigma(\alpha)=\sum_{x\in Q_0}\sigma(x)\alpha(x).$$

In this way, we will identify weights with $\Gamma^\star=\mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})$.

Let us choose the dimension vectors $\alpha=\underline{d}_V$, $\beta=\underline{d}_W$ of V,W in such way that $\langle\alpha,\beta\rangle=0$. Then the matrix of d^V_W in (1) is a square matrix. Following [8] we can therefore define the semi-invariant

$$c(V,W):=\det d^V_W$$

of the action of $\mathrm{GL}(Q,\alpha)\times\mathrm{GL}(Q,\beta)$ on $\mathrm{Rep}(Q,\alpha)\times\mathrm{Rep}(Q,\beta)$. For a fixed V the restriction of c to $\{V\}\times\mathrm{Rep}(Q,\beta)$ defines a semi-invariant c^V in $\mathrm{SI}(Q,\underline{d}_W)$. Schofield proved ([8, Lemma 1.4.]) that the weight of c^V equals $\langle\alpha,\cdot\rangle$. Similarly, for a fixed W the restriction of c to $\mathrm{Rep}(Q,\alpha)\times\{w\}$ defines a semi-invariant c_W in $\mathrm{SI}(Q,\underline{d}_V)$ of weight $-\langle\cdot,\underline{d}_W\rangle$ ([8, Lemma 1.4.]). The main result of [2] is that the semi-invariants of type c^V (resp. c_W) span all the weight spaces in the rings $\mathrm{SI}(Q,\beta)$. Of course the analogous result is true for the semi-invariants c_W.

Remark 1. If $\langle\underline{d}_V,\underline{d}_W\rangle=0$ then we have $c(V,W)=c^V(W)=c_W(V)=0$ if and only if $\mathrm{Hom}_Q(V,W)\neq0$ which is equivalent to $\mathrm{Ext}_Q(V,W)\neq0$.

It was also shown in [2] that $\dim\mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle}=\dim\mathrm{SI}(Q,\alpha)_{-\langle\cdot,\beta\rangle}$. For convenience, we make the following definition.

Definition 2. For dimension vectors α,β with $\langle\alpha,\beta\rangle=0$ we define

$$\alpha\circ\beta:=\dim\mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle}=\dim\mathrm{SI}(Q,\alpha)_{-\langle\cdot,\beta\rangle}.$$

2.3. **Representations in general position.** A representation V is called in general position of dimension α if $V\in\mathrm{Rep}(Q,\alpha)$ lies in a sufficiently small Zariski open subset ("sufficient" here depends on the context). A dimension vector α is called a Schur root if a general representation of dimension α is indecomposable, i.e., not the direct sum of smaller nonzero representations. If α is a Schur root then either $\langle\alpha,\alpha\rangle=1$ and α is called a *real* Schur root, or $\langle\alpha,\alpha\rangle\leq0$ and α is called *imaginary*. We define $\hom_Q(\alpha,\beta)$ is the minimum of all $\dim_K\mathrm{Hom}_Q(V,W)$ with V and W of dimension α and β respectively. Since this dimension depends semicontinously on V and W, we have that $\dim_K\mathrm{Hom}_Q(V,W)=\hom_Q(\alpha,\beta)$ for V, W general of dimension α and β respectively. Similary we define $\mathrm{ext}_Q(\alpha,\beta)$. If $\hom_Q(\alpha,\beta)=\mathrm{ext}_Q(\alpha,\beta)$ we write $\alpha\perp\beta$ and we way say that α is left perpendicular to β. By Remark 1 we have $\alpha\perp\beta$ if and only if $\alpha\circ\beta\neq0$.

We write $\alpha \hookrightarrow \beta$ if a general representation of dimension β contains a subrepresentation of dimension α. We write $\alpha \twoheadrightarrow \beta$ if a general representation of dimension α has a factor of dimension β. The following lemma was proven in [9].

Lemma 3. *We have*

$$\alpha \hookrightarrow \alpha + \beta \quad \Leftrightarrow \quad \operatorname{ext}_Q(\alpha, \beta) = 0 \quad (\Leftrightarrow \alpha + \beta \twoheadrightarrow \beta).$$

For a dimension vector β, $\Sigma(Q, \beta)$ is the set of all weights $\sigma \in \Gamma^\star$ such that $\mathrm{SI}(Q, \beta)_\sigma \neq 0$. From [2] follows that $\Sigma(Q, \beta)$ is the set of all $\sigma \in \Gamma^\star$ such that $\sigma(\beta) = 0$ and $\sigma(\gamma) \leq 0$ for all $\gamma \hookrightarrow \beta$. We will give a necessary and sufficient criterion which γ with $\gamma \hookrightarrow \beta$, correspond to necessary inequalities.

3. Harder-Narasimhan and Jordan-Hölder filtrations

Let us recall some general notions regarding stability. Let Q be a quiver with no oriented cycles. Let us fix a weight $\sigma = -\langle \cdot, \beta \rangle$ where β is a dimension vector. Let α be a dimension vector. A representation V of dimension α is σ-semistable if there exists a semiinvariant $t \in \mathrm{SI}(Q, \alpha)$ of weight $m\sigma$ (m some positive integer) such that $t(V) \neq 0$. It was proved in [2] that the space of semiinvariants of weight $m\sigma$ is spanned by semiinvariants of the form c_W, $W \in \mathrm{Rep}(Q, m\beta)$. Therefore the representation V is σ-semistable if and only if there exists a positive integer m and a representation W of dimension $m\beta$ such that $\mathrm{Hom}_Q(V, W) = \mathrm{Ext}_Q(V, W) = 0$ (see Remark 1). King proves in [4] that the representation V is σ-semistable if and only if for every subrepresentation $W \subset V$ we have $\sigma(\underline{d}_W) \leq 0$.

This notion of stability was related in Section 3 of [7] to the stability defined through a slope of two additive functions. Let us recall this notion of stability.

Definition 4. Let c and r be two additive functions on the abelian category $\mathcal{A}$ such that $r(V) \neq 0$ for any nonzero object V in $\mathcal{A}$. An object V is $(c : r)$-semistable if for every subobject W of V we have

$$\frac{c(W)}{r(W)} \leq \frac{c(V)}{r(V)}.$$

An object V is $(c : r)$-stable if for every subobject W of V we have

$$\frac{c(W)}{r(W)} < \frac{c(V)}{r(V)}.$$

The $(c : r)$-slope of the object V is the ratio $\mu(V) = \frac{c(V)}{r(V)}$. Then Rudakov shows that the order $\prec$ defined by

$$V \prec W \Leftrightarrow \mu(V) < \mu(W)$$

and equivalence relation defined by

$$V \asymp W \Leftrightarrow \mu(V) = \mu(W)$$

defines a stability order in the sense of [7], Section 1.

In our application we take $\mathcal{A} = \mathrm{Rep}_K(Q)$, $c(V) = \sigma(\underline{d}_V)$ and $r(V) = \tau(\underline{d}_V)$ with $\tau(\alpha) \neq 0$ for all nonzero dimension vectors α (for example $\tau(\alpha) = \sum_{x \in Q_0} \alpha(x)$).

The Theorems 2 and 3 from [7] give immediately the following results.

Proposition 5. *With the assumptions of Definition 4 we have:*

1. *(Harder-Narasimhan filtration) Every object V has a filtration*

$$V = F_H^0(V) \supset F_H^1(V) \supset \ldots \supset F_H^m(V) \supset F_H^{m+1}(V) = 0$$

such that
(a) *each factor $G_H^i(V) = F_H^i(V)/F_H^{i+1}(V)$ is $(c:r)$-semistable;*
(b)

$$G_H^0(V) \prec G_H^1(V) \prec \ldots \prec G_H^m(V).$$

The filtration with properties (a), (b) is unique,

2. *(Jordan-Hölder filtration) Every $(c:r)$-semistable object V has a filtration*

$$V = F_J^0(V) \supset F_J^1(V) \supset \ldots \supset F_J^m(V) \supset F_J^{m+1}(V) = 0$$

such that
(a) *each factor $G_J^i(V) = F_J^i(V)/F_J^{i+1}(V)$ is $(c:r)$-stable;*
(b)

$$G_J^0(V) \asymp G_J^1(V) \asymp \ldots \asymp G_J^m(V).$$

The set of factors $\{G_J^i(V)\}$ is uniquely determined by the properties (a), (b).

Remark 6. Of course, by taking the Harder-Narasimhan filtration and using the Jordan-Hölder filtration on each quotient, one obtains a unique filtration of an object V such that every quotient is $(c:r)$-*stable*. This filtration we will call Harder-Narasimhan-Jordan-Hölder filtration or HNJH-filtration for short.

Lemma 7. *Let Q be a quiver and $\alpha_0 = \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1} = 0$ are dimension vectors. The set of all $V \in \mathrm{Rep}(Q, \alpha)$ of representations which allow a filtration*

$$V = F^0(V) \supset F^1(V) \supset \cdots \supset F^{m+1}(V) = 0$$

with $\underline{d}_{F^i(V)} = \alpha_i$ is Zariski closed.

Proof. The proof goes exactely as in [9, Section 3]. □

Proposition 8. *Let Q be a quiver without oriented cycles, σ, τ ($\tau(\alpha) > 0$ for all nonzero dimension vectors) be weights and α be a dimension vector.*

1. *There exists an nonempty open set $U \subset \mathrm{Rep}_K(Q, \alpha)$ such that for $V \in U$ the dimensions of the factors of the Harder-Narasimhan filtration with respect to $(\sigma : \tau)$ of V are constant.*
2. *There exists an nonempty open set $U \subset \mathrm{Rep}_K(Q, \alpha)$ such that for $V \in U$ the dimensions of the factors of the HNJH-filtration with respect to $(\sigma : \tau)$ of V are constant.*
3. *If a general representation of dimension α is σ-stable, then there exists an nonempty open set $U \subset \mathrm{Rep}_K(Q, \alpha)$ such that for $V \in U$ the dimensions of the factors of the Jordan-Hölder-filtration of V are constant.*

Proof. This is an obvious consequence of the definitions and of the previous lemma. □

4. σ-STABLE DECOMPOSITION

Definition 9. Let Q be a quiver without oriented cycles, α be a dimension vectors and let σ, τ be weights We assume that τ is positive on nonzero dimension vectors. We call the dimension vector α σ-stable (resp. $(\sigma : \tau)$-stable) if a general representation of dimension α is σ-stable (resp. $(\sigma : \tau)$-stable). The expression

$$\alpha = \alpha_1 \dotplus \alpha_2 \dotplus \ldots \dotplus \alpha_s$$

is called the *σ-stable decomposition* (resp. *$(\sigma : \tau)$-stable decomposition*) of α if a general representation V of dimension α has a Jordan-Hölder filtration (resp. HNJH-filtration) with factors of dimension $\alpha_1, \alpha_2, \dots, \alpha_s$ (in some order). We may use the abbreviation $c \cdot \beta$ instead of $\beta \dotplus \beta \dotplus \cdots \dotplus \beta$ (c copies).

Lemma 10. *Suppose that α has $(\sigma : \tau)$-stable decomposition*

$$\alpha_1 \dotplus \alpha_2 \dotplus \cdots \dotplus \alpha_s.$$

For any r the $(\sigma : \tau)$-stable decomposition of $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ is

$$\alpha_1 \dotplus \alpha_2 \dotplus \cdots \dotplus \alpha_r.$$

A similar statement is true for σ-stable decomposition.

Proof. Suppose that the $(\sigma : \tau)$-stable decomposition of β is

$$\beta_1 \dotplus \beta_2 \dotplus \cdots \dotplus \beta_t.$$

Let V_i be a $(\sigma : \tau)$-stable representation of dimension α_i for all $i \leq s$ and let W be a general representation of dimension β. The representation $V_1 \oplus V_2 \oplus \cdots \oplus V_r$ of dimension β has a HNJH-filtration with r nonzero quotients (namely $V_1, \dots, V_r$). Since W is in general position its HNJH-filtration cannot be longer by Lemma 7, therefore $t \leq r$. Put

$$Z = W \oplus V_{r+1} \oplus \cdots \oplus V_s.$$

Any HNJH-filtration of Z has quotients of dimensions $\beta_1, \dots, \beta_t, \alpha_{r+1}, \dots, \alpha_s$.

By Lemma 7, Z also has a filtration

$$0 \subset F^0(Z) \subset F^1(Z) \subset \cdots \subset F^s(Z) = Z$$

with quotients of dimensions $\alpha_1, \alpha_2, \dots, \alpha_s$ in some order. The quotients $F^i(Z)/F^{i-1}(Z)$ are σ-semistable. We can get a HNJH-filtration of Z by refining the filtration $\{F^i(Z)\}$. Since any HNJH-filtration of Z has $t + s - r \geq s$ nonzero quotients, we must have that $t = r$, and $\{F^i(Z)\}$ is a HNJH-filtration. We conclude that $\beta_1, \dots, \beta_t, \alpha_{r+1}, \dots, \alpha_s$ is a permutation of $\alpha_1, \dots, \alpha_s$, so $\beta_1, \dots, \beta_t$ is a permutation of $\alpha_1, \dots, \alpha_r$.

Notice that if τ is any weight with $\tau(\alpha) > 0$ for all nonzero dimension vectors, and α is σ-stable, then the σ-stable decomposition of α is the same as the $(\sigma : \tau)$-stable decomposition of α. □

In the next proposition we list the basic properties of σ-stable decompositions.

Proposition 11. *Let Q, σ, α be as above. Let*

$$\alpha = c_1 \cdot \alpha_1 \dotplus c_2 \cdot \alpha_2 \dotplus \cdots \dotplus c_s \cdot \alpha_s$$

be the σ-stable decomposition where the α_i are distinct.

(a). *All α_i are Schur roots;*
(b). *if $c_i > 1$ then α_i must be a real or an isotropic root;*
(c). $\hom_Q(\alpha_i, \alpha_j) = 0$ *if $i \neq j$;*
(d). *after rearranging one can assume that* $\operatorname{ext}_Q(\alpha_i, \alpha_j) = 0$ *for all $i < j$;*
(e). *under the assumptions of (d), $(p\alpha_i) \circ (q\alpha_j) = 1$ for all $i < j$ and all nonnegative integers p, q.*

Proof. (a) If general representations of dimension α_i are decomposable, then a general representation of dimension α_i is not σ-stable. It follows that there are no σ-stable representations in dimension α_i.

(b) If $c_i \geq 2$, then by Lemma 10, the σ-stable decomposition of $2\alpha_i$ is $\alpha_i \dotplus \alpha_i$. In particular we have $\alpha_i \hookrightarrow 2\alpha_i$, so $\operatorname{ext}_Q(\alpha_i, \alpha_i) = 0$ and $\langle \alpha_i, \alpha_i \rangle \geq 0$.

(c) Let V_i and V_j be σ-stable representations of dimensions α_i and α_j respectively. From [7, Theorem 1] follows that any nonzero homomorphism between V_i and V_j must be an isomorphism. Since $\alpha_i \neq \alpha_j$, we have $\operatorname{Hom}_Q(V_i, V_j) = 0$ and therefore $\hom_Q(\alpha_i, \alpha_j) = 0$.

(d) Assume that the statement in (d) is false. Then there exists an $r \geq 2$ such that after rearranging we have an r-cycle:

$$\operatorname{ext}_Q(\alpha_1, \alpha_2) \neq 0,\ \operatorname{ext}_Q(\alpha_2, \alpha_3) \neq 0, \ldots,\ \operatorname{ext}_Q(\alpha_{r-1}, \alpha_r) \neq 0,\ \operatorname{ext}_Q(\alpha_r, \alpha_1) \neq 0.$$

We assume that $r \geq 2$ is minimal such that an r-cycle exists, so $\operatorname{ext}_Q(\alpha_i, \alpha_j) = 0$ unless $j = i$ or $j = i + 1$ (modulo r). The σ-stable decomposition of $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ is

$$\alpha_1 \dotplus \alpha_2 \dotplus \cdots \dotplus \alpha_r.$$

For some i we have $\alpha_i \hookrightarrow \beta$, and after reordering we may assume that $\alpha_1 \hookrightarrow \beta$, so $\operatorname{ext}_Q(\alpha_1, \beta - \alpha_1) = 0$. Now for all $i \geq 3$ we have $\operatorname{ext}_Q(\alpha_i, \alpha_2) = 0$. It follows that $\operatorname{ext}_Q(\alpha_3 + \alpha_4 + \cdots + \alpha_r, \alpha_2) = 0$ or equivalently, $\beta - \alpha_1 \twoheadrightarrow \alpha_2$. Consider an exact sequence

$$0 \to V' \to V \to V'' \to 0 \tag{3}$$

with V', V, V'' of dimension $\beta - \alpha_1 - \alpha_2, \beta - \alpha_1, \beta_2$ respectively, and V is in general position. Let W be a general representation of dimension α_1 and apply the functor $\operatorname{Hom}_Q(W, \cdot)$ to (3) to obtain a long exact sequence

$$\cdots \to \operatorname{Ext}_Q(W, V') \to \operatorname{Ext}_Q(W, V) \to \operatorname{Ext}_Q(W, V'') \to 0.$$

Since $\operatorname{Ext}_Q(W, V) = 0$, we have $\operatorname{Ext}_Q(W, V'') = 0$, so $\operatorname{ext}_Q(\alpha_1, \alpha_2) = 0$. Contradiction.

(e) Let $U \subseteq \operatorname{Rep}(Q, p\alpha_i)$ be an open set of σ-semistable representations whose Jordan-Hölder filtration has only factors whose dimension is proportional to α_i. Let $U' \subseteq \operatorname{Rep}(Q, q\alpha_j)$ be the open set of σ-semistable representations in dimension $q\alpha_j$ whose Jordan-Hölder filtration has only factors whose dimension is proportional to α_j. Let V be a generic representation of dimension $q\alpha_i$. Then the semiinvariant $c^V \in \operatorname{SI}(Q, q\alpha_j)$ does not vanish on U'. Indeed, let $W \in U'$ be a representation such that $c^V(W) = 0$. Then $\operatorname{Hom}_Q(V, W) \neq 0$ which contradicts Theorem 1 from [7]. Thus for every V in U the zero set of c^V is contained in the complement of U'. Therefore this zero set does not depend on V and the space of semiinvariants $\operatorname{SI}(Q, q\alpha_j)_{\langle p\alpha_i, \cdot \rangle}$ is one-dimensional as claimed. $\square$

Corollary 12. *Let Q, σ, τ, α be as above. Let*

$$\alpha = c_1 \cdot \alpha_1 \dotplus c_2 \cdot \alpha_2 \dotplus \cdots \dotplus c_s \cdot \alpha_s$$

be the $(\sigma : \tau)$-stable decomposition where the α_i are distinct.

(a). *All α_i are Schur roots;*
(b). *if $c_i > 1$ then α_i must be a real or an isotropic root;*
(c). *after rearranging one can assume that $\alpha_i \perp \alpha_j$ for all $i < j$;*
(d). *under the assumptions of (d), $(p\alpha_i) \circ (q\alpha_j) = 1$ for all $i < j$ and all nonnegative integers p, q.*

Proof. (a) and (b) follow immediately from Proposition 11 (a), (b). Let $\mu = \sigma/\tau$. We may assume that $\mu(\alpha_i) \geq \mu(\alpha_j)$ for all $i < j$. We have $\operatorname{Hom}_Q(\alpha_i, \alpha_j) = 0$ for $i < j$. Suppose that $\mu(\alpha_i) > \mu(\alpha_j)$. The $(\sigma : \tau)$-stable decomposition of $\alpha_i + \alpha_j$ is $\alpha_i \dotplus \alpha_j$. We have either $\alpha_i \hookrightarrow \alpha_i + \alpha_j$ or $\alpha_j \hookrightarrow \alpha_i + \alpha_j$, but because $\mu(\alpha_i) > \mu(\alpha_j)$ we have $\alpha_i \hookrightarrow \alpha_i + \alpha_j$. This shows that $\operatorname{ext}_Q(\alpha_i, \alpha_j)$ for all i, j with $\mu(\alpha_i) > \mu(\alpha_j)$. By Proposition 11 (d) we may assume that for all $i < j$ we have $\operatorname{ext}_Q(\alpha_i, \alpha_j)$. This proves (c). The prove of (d) is similar to the proof of Proposition 11 (e). $\square$

Theorem 13. *Suppose that σ is an indivisible weight. If $\alpha = c_1 \cdot \alpha_1 \dotplus c_2 \cdot \alpha_2 \dotplus \cdots \dotplus c_r \cdot \alpha_r$ is the σ-stable decomposition of α, then there exists an isomorphism*

$$\operatorname{SI}(Q, \alpha)_{m\sigma} \cong S^{c_1}(\operatorname{SI}(Q, \alpha_1)_{m\sigma}) \otimes S^{c_2}(\operatorname{SI}(Q, \alpha_2)_{m\sigma}) \otimes \cdots \otimes S^{c_r}(\operatorname{SI}(Q, \alpha_r)_{m\sigma})$$

Proof. Let

$$S := \operatorname{Rep}(Q, \alpha_1)^{c_1} \oplus \operatorname{Rep}(Q, \alpha_2)^{c_2} \oplus \cdots \oplus \operatorname{Rep}(Q, \alpha_r)^{c_r}.$$

We have a natural embedding

$$\varphi : S \hookrightarrow \operatorname{Rep}(Q, \alpha).$$

Let G be the stabilizer of S within $\operatorname{GL}(\alpha)_\sigma$. This group G is isomorphic to the intersection of

$$(S_{c_1} \ltimes \operatorname{GL}(\alpha_1)^{c_1}) \times (S_{c_2} \ltimes \operatorname{GL}(\alpha_2)^{c_2}) \times \cdots \times (S_{c_r} \ltimes \operatorname{GL}(\alpha_r)^{c_r})$$

and $\operatorname{GL}(\alpha)_\sigma$. Here S_c is the symmetric group on c elements. Let $\pi_S : S \to S/\!/G$ and $\pi : \operatorname{Rep}(Q, \alpha) \to \operatorname{Rep}(Q, \alpha)/\!/\operatorname{GL}(\alpha)_\sigma$ be the categorical quotients. The embedding φ induces a morphism between categorical quotients

$$\psi : S/\!/G \to \operatorname{Rep}(Q, \alpha)/\!/\operatorname{GL}(\alpha)_\sigma.$$

We will show that ψ is an isomorphism.

First we will show that ψ is dominant. Let $V \in \operatorname{Rep}(Q, \alpha)$ be a general representation and suppose V is σ-stable, i.e., $\pi(V) \neq 0$. Let

$$0 = F_J^0(V) \subset F_J^1(V) \subset \cdots \subset F_J^s(V) = V$$

be a Jordan-Hölder filtration with σ-stable quotients $G^i(V) = F^i(V)/F^{i-1}(V)$. Now $W := \bigoplus_i G^i(V) \in S$ lies in the $\operatorname{GL}(\alpha)_\sigma$ closure of V. In particular $\psi(\pi_S(W)) = \pi(V)$. This proves that ψ is surjective.

Note that $W \in S$ is G-semistable if and only if all the summands in $\operatorname{Rep}(Q, \alpha_i)$ are σ-semistable. In particular, if $W \in S$ is G-semistable, then W σ-semistable, so W is $\operatorname{GL}(\alpha)_\sigma$-stable. This shows that $\psi^{-1}(0) = \{0\}$, so ψ is a finite map.

For a general representation $V \in \operatorname{Rep}(Q, \alpha)$, the quotients of the Jordan-Hölder filtration corresponding to σ are unique up to permutation. This shows that a generic fiber of ψ consists of only one point. So ψ is birational.

Because ψ is birational and finite, and $\operatorname{Rep}(Q, \alpha)/\!/\operatorname{GL}(\alpha)_\sigma$ is normal, ψ must be an isomorphism. Now the graded coordinate ring of $\operatorname{Rep}(Q, \alpha)/\!/\operatorname{GL}(\alpha)_\sigma$ is $\bigoplus_{m \geq 0} \operatorname{SI}(Q, \alpha)_{m\sigma}$ and $S/\!/G$ has graded coordinate ring

$$\bigoplus_{m \geq 0} \operatorname{SI}(Q, \alpha)_{m\sigma} \cong S^{c_1}(\operatorname{SI}(Q, \alpha_1)_{m\sigma}) \otimes S^{c_2}(\operatorname{SI}(Q, \alpha_2)_{m\sigma}) \otimes \cdots \otimes S^{c_r}(\operatorname{SI}(Q, \alpha_r)_{m\sigma}).$$

$\square$

5. Schur sequences

Proposition 11 motivates us for the definition of Schur sequences. The notion of Schur sequences is closely related to the notion of exceptional sequences (see [1]).

Definition 14. A sequence $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s)$ is called an exceptional sequence if

1. ε_i is a real Schur root for every i;
2. $\varepsilon_i \perp \varepsilon_j$ for all $i < j$.

A Schur sequence is similar to an exceptional sequence, but also imaginary Schur roots are allowed.

Definition 15. A sequence $\underline{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_s)$ is called a Schur sequence if

1. γ_i is a Schur root for every i;
2. $p\gamma_i \circ q\gamma_j = 1$ for all $i < j$ and all positive integers p, q (in particular, $\gamma_i \perp \gamma_j$).

Lemma 16. *If* $\operatorname{Rep}(Q,\alpha)$ *has a dense* $\operatorname{GL}(\alpha)$*-orbit or* $\operatorname{Rep}(Q,\beta)$ *has a dense* $\operatorname{GL}(\beta)$*-orbit, then* $\alpha \circ \beta \leq 1$.

Proof. If $\operatorname{Rep}(Q,\beta)$ has a dense $\operatorname{GL}(\beta)$-orbit, then there are no rational $\operatorname{GL}(\beta)$-invariants in $K[\operatorname{Rep}(Q,\beta)]$. In particular, any quotient of two semiinvariants of the same weight must be constant. This shows that

$$\alpha \circ \beta = \dim_K \operatorname{SI}(Q,\beta)_{\langle \alpha, \cdot \rangle} \leq 1.$$

If $\operatorname{Rep}(Q,\alpha)$ has a dense $\operatorname{GL}(\alpha)$-orbit then the proof is similar. □

Lemma 17. *Suppose that* $\alpha \perp \gamma$ *and* $\beta \perp \gamma$.

1. *If* $(\alpha + \beta) \circ \gamma = 1$*, then* $\alpha \circ \gamma = 1$ *and* $\beta \circ \gamma = 1$*;*
2. *if* $\operatorname{ext}_Q(\alpha,\beta) = 1$*,* $\alpha \circ \gamma = 1$ *and* $\beta \circ \gamma = 1$ *then* $(\alpha+\beta) \circ \gamma = 1$.

Proof. (a) Choose $f \in \operatorname{SI}(Q,\gamma)_{\langle \alpha, \cdot \rangle}$. Then we have

$$f \operatorname{SI}(Q,\gamma)_{\langle \beta, \cdot \rangle} \subseteq \operatorname{SI}(Q,\gamma)_{\langle \alpha+\beta, \cdot \rangle}.$$

This shows that $\beta \circ \gamma \leq (\alpha+\beta) \circ \gamma$. Similarly we have $\alpha \circ \gamma \leq (\alpha+\beta) \circ \gamma$.

(b) Any $(\alpha+\beta)$-dimensional representation V has an α-dimensional subrepresentation V'. If $V'' = V/V'$ then for any $W \in \operatorname{Rep}(Q,\gamma)$ we have $c^V(W) = c^{V'}(W)c^{V''}(W)$. The lemma follows from the fact that $\operatorname{SI}(Q,\gamma)_{\langle \alpha+\beta, \cdot \rangle}$, $\operatorname{SI}(Q,\gamma)_{\langle \alpha, \cdot \rangle}$ and $\operatorname{SI}(Q,\gamma)_{\langle \beta, \cdot \rangle}$ are spanned by c^V's, $c^{V'}$'s and $c^{V''}$'s respectively. □

Corollary 18. *If* $\gamma_1, \gamma_2, \dots, \gamma_s$ *is a Schur sequence, and* $p\gamma_i + q\gamma_{i+1}$ *is a Schur root, then*

$$\gamma_1, \gamma_2, \dots, \gamma_{i-1}, p\gamma_i + q\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_s$$

is a Schur sequence.

Remark 19. An exceptional sequence $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s)$ is a Schur sequence. The space $\operatorname{Rep}(Q, q\varepsilon_j)$ has a dense $\operatorname{GL}(q\varepsilon_j)$-orbit, so by Lemma 16

$$p\varepsilon_i \circ q\varepsilon_j = \dim \operatorname{SI}(Q, q\varepsilon_j)_{\langle p\varepsilon_i, \cdot \rangle} \leq 1.$$

Remark 20. Suppose that $\alpha = \alpha_1^{\oplus c_1} \oplus \alpha_2^{\oplus c_2} \oplus \cdots \oplus \alpha_s^{\oplus c_s}$ is the canonical decomposition (notation as in [3]) with all α_i distinct. It was proven in [6] that $\operatorname{ext}_Q(\alpha_i, \alpha_j) = 0$ for all $i \neq j$. In [9], it was proven that after reordering we may assume that $\hom_Q(\alpha_i, \alpha_j) = 0$ for all $i < j$. We claim that in fact $\alpha_1, \alpha_2, \dots, \alpha_s$ is a Schur sequence. This follows from the algorithm in [3] for finding the canonical decomposition and Corollary 18. In [3, Definition 5], a sequence $\alpha_1, \dots, \alpha_s$ was called a compartment if

1. all α_i are Schur roots;
2. $\alpha_i \perp \alpha_j$ for all $i < j$;
3. $\langle \alpha_j, \alpha_i \rangle \geq 0$ for all $i < j$.

If all α_i are distinct, then the canonical decomposition of $\alpha_1 + \alpha_2 + \cdots + \alpha_s$ is $\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_s$ by [3, Proposition 7]. This shows that compartments are Schur sequences.

Corollary 21. *Suppose that $\alpha = c_1 \cdot \alpha_1 \dotplus c_2 \cdot \alpha_2 \dotplus \cdots \dotplus c_r \cdot \alpha_r$ is a $(\sigma : \tau)$-stable decomposition. By Corollary 12 (c) we may assume that $\alpha_i \perp \alpha_j$ for all $i < j$. By Corollary 12 (a) and (d), $\alpha_1, \alpha_2, \ldots, \alpha_r$ is a Schur sequence.*

Definition 22. A Schur sequence $\underline{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_s)$ is called a quiver Schur sequence if $\langle \gamma_j, \gamma_i \rangle \leq 0$ for all $i < j$.

Corollary 23. *Suppose that $\alpha = c_1 \cdot \alpha_1 \dotplus c_2 \cdot \alpha_2 \dotplus \cdots \dotplus c_r \cdot \alpha_r$ is a σ-stable decomposition. By Proposition 11 (d) we may assume that $\operatorname{ext}_Q(\alpha_i, \alpha_j) = 0$ for all $i < j$. By Proposition 11 (a), (c) and (e), $\alpha_1, \alpha_2, \ldots, \alpha_r$ is a quiver Schur sequence.*

Definition 24. Let $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$, $\underline{\beta} = (\beta_1, \ldots, \beta_s)$ be two sequences of dimension vectors. We say that $\underline{\beta}$ is a refinement of $\underline{\gamma}$ if there exists a sequence $0 = b_0 < b_1 < \ldots < b_{r-1} < s = b_r$ such that for each $j = 1, \ldots, r$ the dimension vector γ_j is a positive linear combination of $\beta_{b_{j-1}+1}, \ldots, \beta_{b_j}$.

Theorem 25. *Let $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$ be a Schur sequence. Then there exists an exceptional sequence $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_s)$ such that $\underline{\varepsilon}$ is a refinement of $\underline{\gamma}$.*

Proof. For a dimension vector α we define $\mu(\alpha) = \sum_{x \in Q_0} \alpha(x)$. We will prove the theorem by induction on the number of vertices n in the quiver Q, and by induction on $\mu(\gamma_1)$. If $n = 1$ there is nothing to prove. If $\mu(\gamma_1) = 0$ then there is nothing to prove either since this is impossible.

Let us assume that in a Schur sequence the first dimension vector γ_1 is a real Schur root. Then the dimension vectors $\gamma_2, \ldots, \gamma_r$ are Schur roots in the right orthogonal category $\gamma_1^\perp$. By [9, Theorem 2.3] the category $\gamma_1^\perp$ is equivalent to the category of representations of a quiver Q' with no oriented cycles and $n - 1$ vertices. The sequence $\underline{\gamma'} = (\gamma_2, \ldots, \gamma_r)$ will be a Schur sequence in this category. By induction we can refine it to the exceptional sequence $\underline{\varepsilon'} = (\varepsilon_2, \ldots, \epsilon_s)$. Then the sequence $\underline{\varepsilon} = (\gamma_1, \varepsilon_2, \ldots, \varepsilon_n)$ is clearly an exceptional sequence for Q which refines $\underline{\gamma}$.

Let us now assume that γ_1 is an imaginary root. Notice that

$$\operatorname{ext}_Q(p\gamma_i, p(\gamma_{i+1} + \cdots + \gamma_r)) = 0$$

for all i. Since $q\gamma_1 \circ p\gamma_j = 1$ for all $j \geq 2$, it follows by induction from Lemma 17 (a) that $q\gamma_1 \circ p(\gamma_i + \cdots + \gamma_r) = 1$ for all $i > 1$. Put $\delta = \gamma_2 + \cdots + \gamma_r$. Then $q\gamma_1 \circ p\delta = 1$ for all positive integers p, q.

Let $\gamma_1^\perp$ be the set of all dimension vectors α with $\gamma_1 \perp \alpha$. By Theorem 2 from [2], $\gamma_1^\perp$ is a rational polyhedral cone in the space of all dimension vectors. Suppose that δ is in the interior of the cone. For each $\alpha \in \gamma_1^\perp$ there exists $\beta \in \gamma_1^\perp$ such that $\alpha + \beta = p\delta$ for some positive integer p. From Lemma 17 (b) follows that $\gamma \circ \alpha = 1$. This shows that for all σ, $\dim_K \operatorname{SI}(Q, \gamma_1)_\sigma = 1$. So there are no rational invariants and γ_1 must be a *real* Schur root. Contradiction, so it follows that δ is not in the interior of $\gamma_1^\perp$.

Let $\sigma = -\langle\cdot,\delta\rangle$ and let us study the σ-stable decomposition of γ_1. By [2, Theorem 2], there exists a $\beta \hookrightarrow \gamma_1$ such that $\sigma(\beta) = 0$. In particular, the σ-stable decomposition of γ_1 is nontrivial. Suppose that

$$\gamma_1 = c_1 \cdot \beta_1 \dotplus c_2 \cdot \beta_2 \dotplus \cdots \dotplus c_l \cdot \beta_l$$

is the σ-stable decomposition of γ_1. We may assume that $\beta_i \perp \beta_j$ for $i < j$. Then it is easy to check that

$$\underline{\gamma}' = (\beta_1, \beta_2, \ldots, \beta_l, \gamma_2, \ldots, \gamma_r) \tag{4}$$

is a Schur sequence using Lemma 17 (b). Notice that β_1 is smaller than γ_1. Now $\mu(\beta_1) < \mu(\gamma_1)$ so by induction there exists an exceptional sequence which is a refinement of $\underline{\gamma}'$. ☐

Corollary 26. *Let $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$ be a Schur sequence. Then the vectors $\gamma_1, \ldots, \gamma_r$ are linearly independent.*

Proof. The vectors occurring in an exceptional sequence are linearly independent. ☐

6. The faces of the cone $\Sigma(Q,\alpha)$

The Refinement Theorem allows us to obtain a beautiful description of the faces of the cone $\Sigma(Q,\alpha)$. Let us denote by $W_r(Q,\alpha)$ the set of quiver Schur sequences $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$ of length r such that $\alpha = \sum_{i=1}^r a_i\gamma_i$ with

1. a_i a positive integer for all i;
2. if γ_i is imaginary and not isotropic, then $a_i = 1$.

If one quiver Schur sequence is a permutation of another quiver Schur sequence, they are considered the same. Let $F_r(Q,\alpha)$ be the set of faces of dimension $n - r$ of $\Sigma(Q,\alpha)$.

Theorem 27. *Let Q be a quiver without oriented cycles and let α be a dimension vector. For each r, $1 \leq r \leq n-1$ there is a natural bijection*

$$\psi(r) : W_r(Q,\alpha) \to F_r(Q,\alpha)$$

which sends the quiver Schur sequence $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$ to the face

$$\Sigma(Q,\gamma_1) \cap \ldots \cap \Sigma(Q,\gamma_r).$$

The inverse bijection is obtained as follows. For a given face F we take the weight σ in the relative interior of F and associate to it the quiver Schur sequence coming from the σ-stable decomposition of α.

Proof. Let $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$ be a quiver Schur sequence such that $\alpha = \sum_{i=1}^r a_i\gamma_i$ with $a_i > 0$ and $a_i = 1$ whenever γ_i is imaginary and nonisotropic.

Let us prove first that $\Sigma(Q,\gamma_1) \cap \ldots \cap \Sigma(Q,\gamma_r)$ is a face of codimension r in the space of dimension vectors. By the Refinement Theorem there exists an exceptional sequence $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_s)$ which is a refinement of $\underline{\gamma}$, i.e., there exists a sequence $0 = b_0 < b_1 < \ldots < b_{r-1} < n = b_r$ such that for each $j = 1, \ldots, r$ the dimension vector γ_j is a positive linear combination of $\varepsilon_{b_{j-1}+1}, \ldots, \varepsilon_{b_j}$.

We proceed by induction on r. For $r = 1$ there is nothing to prove because γ_1 is a Schur root and $\Sigma(Q,\gamma_1)$ has dimension $n - 1$.

Suppose that $r > 1$. The category spanned by $\varepsilon_1, \ldots, \varepsilon_{b_1}$ is the category of representations of a quiver Q' with b_1 vertices and without oriented cycles. The right orthogonal category $\varepsilon_1^\perp \cap \cdots \cap \varepsilon_{b_1}^\perp$ is the category of representations of a quiver Q'' with $n-b_1$ vertices and without oriented cycles. Let us define $\Gamma' = \mathrm{Hom}(Q_0', \mathbb{Z})$ and $\Gamma'' = \mathrm{Hom}(Q_0'', \mathbb{Z})$ and notice that $\Gamma = \Gamma' \oplus \Gamma''$ and $\Gamma^\star = (\Gamma')^\star \oplus (\Gamma'')^\star$.

By induction hypothesis, we can find linearly independent

$$\sigma_1', \dots, \sigma_{b_1-1}' \in \Sigma(Q', \gamma_1) \subset (\Gamma')^\star \subset \Gamma^\star$$

and linearly independent

$$\sigma_1'', \dots, \sigma_{n-b_1-r+1}'' \in \Sigma(Q'', \gamma_2) \cap \cdots \cap \Sigma(Q'', \gamma_r) \subset (\Gamma'')^\star \subset \Gamma^\star$$

Now

$$\sigma_1', \dots, \sigma_{b_1-1}', \sigma_1'', \dots, \sigma_{n-b_1-r+1}'' \in \Sigma(Q, \gamma_1) \cap \cdots \cap \Sigma(Q, \gamma_r)$$

are $n-r$ linearly independent weights. This shows that $\Sigma(Q,\gamma_1) \cap \cdots \cap \Sigma(Q,\gamma_r)$ has dimension $n-r$.

We have proven that the map $\psi(r)$ defined above is well defined. Let us show that the inverse map is well defined and that it is indeed an inverse. Let $\mathcal{F}$ be a face of dimension $n-r$ of $\Sigma(Q,\alpha)$. Take a dimension vector σ in the relative interior of $\mathcal{F}$. Let

$$\alpha = c_1 \cdot \delta_1 \dotplus c_2 \cdot \delta_2 \dotplus \cdot \dotplus c_l \cdot \delta_l$$

be the σ-stable decomposition of α. Define $\mathcal{F}' = \Sigma(Q,\delta_1) \cap \cdots \cap \Sigma(Q,\delta_l)$. Since $\sigma \in \mathcal{F}'$ we have $\mathcal{F} \subseteq \mathcal{F}'$.

Let $\gamma \hookrightarrow \alpha$, $\sigma(\gamma) = 0$. Then γ is a linear combination of δ_i's by the definition of σ-stable decomposition. But the description of $\Sigma(Q,\alpha)$ given in [2, Theorem 1] implies that

$$\mathcal{F} = \bigcap_{\gamma, \gamma \hookrightarrow \alpha, \langle \gamma, \beta \rangle = 0} \{\langle \gamma, \cdot \rangle = 0\}$$

so we have $\mathcal{F}' \subseteq \mathcal{F}$. This concludes the proof of the Theorem. □

Let us state the meaning of Theorem 2 in two extreme cases: for the walls of maximal dimension of $\Sigma(Q,\alpha)$ and for extremal rays.

Corollary 28. *Let Q be a quiver without oriented cycles and let α be a Schur root. Then the walls of $\Sigma(Q,\alpha)$ (i.e., faces of dimension $n-1$) are in one to one correspondance with the ways of writing*

$$\alpha = c_1\gamma_1 + c_2\gamma_2$$

where (γ_1, γ_2) is a quiver Schur sequence, c_1, c_2 positive integers with $c_i = 1$ whenever γ_i is imaginary and nonisotropic.

Let us consider an extremal ray σ in $\Sigma(Q,\alpha)$. It corresponds to the linear combination

$$\alpha = c_1\gamma_1 + \ldots + c_{n-1}\gamma_{n-1}$$

where $(\gamma_1, \dots, \gamma_{n-1})$ is a quiver Schur sequence. The Refinement Theorem implies that $n-2$ of the roots $\gamma_1, \dots, \gamma_{n-1}$ are real Schur roots. Consider the subring

$$\mathrm{SI}(Q,\alpha,\sigma) = \bigoplus_{m \geq 0} \mathrm{SI}(Q,\alpha)_{m\sigma}$$

By peeling off real roots from the left and from the right we can reduce the calculation of this ring to the ring of semi-invariants for a quiver with two vertices.

Corollary 29. *Let Q be a quiver with no oriented cycles and let α be a Schur root. Let σ be an extremal ray. Then there exists an n and a Schur root β for the quiver $\theta(n)$ such that*

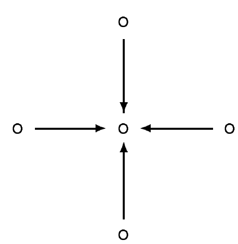
$$\mathrm{SI}(Q, \alpha, \sigma) \cong \mathrm{SI}(\theta(n), \gamma).$$

The description of the walls has still one drawback. One would like to replace the condition $p\gamma_i \circ q\gamma_j = 1$ for all positive integers p, q, just by $\gamma_i \circ \gamma_j = 1$. This seems to be equivalent. Let us state it as a conjecture.

Conjecture 30. *Let Q be a quiver without oriented cycles. Let α be a dimension vector and σ be a weight. Assume that* $\dim \mathrm{SI}(Q, \alpha)_\sigma = 1$. *Then for all positive integers p, q we have* $\dim \mathrm{SI}(Q, p\alpha)_{q\sigma} = 1$.

For the triple flag quiver this is equivalent to a conjecture of Fulton. In that case Knutson, Tao and Woodward announced that they could prove this conjecture.

Example 31. Let Q be the quiver



and let α be the dimension vector

$$\begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{array}$$

There are 8 walls of $\Sigma(Q, \alpha)$. They are given by the Schur sequences

$$\begin{array}{ccc} & 0 & \\ 1 & 2 & 1 \\ & 1 & \end{array}, \quad \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \qquad \text{(4 by symmetry)}$$

$$\begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{array}, \quad \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \qquad \text{(4 by symmetry).}$$

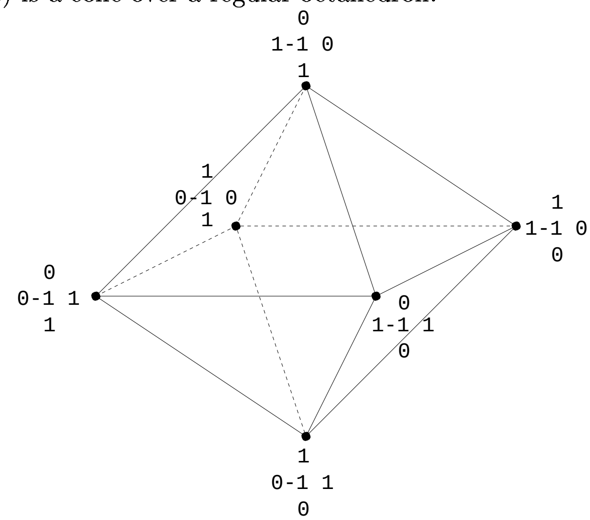
There are 12 two-dimensional faces of the cone given by the sequences

$$\begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{array}, \quad \begin{array}{ccc} & 0 & \\ 1 & 1 & 0 \\ & 1 & \end{array}, \quad \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \qquad \text{(12 by symmetry).}$$

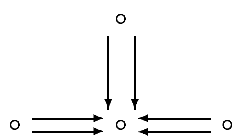
There are 6 extremal rays, given by the Schur sequences

$$\begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{array}, \quad \begin{array}{ccc} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{array}, \quad \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{array}, \quad \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \qquad \text{(6 by symmetry).}$$

The set $\Sigma(Q,\alpha)$ is a cone over a regular octahedron.



Example 32. Let Q be the quiver



and let α be the dimension vector

$$\begin{matrix} & 1 & \\ 1 & 3 & 1 \end{matrix}.$$

Now $\Sigma(Q,\alpha)$ has 6 walls corresponding to the Schur sequences

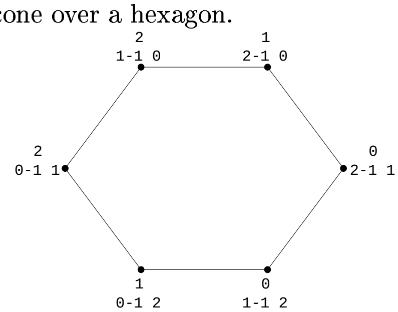
$$\begin{matrix} & 0 & \\ 1 & 3 & 1 \end{matrix}, \begin{matrix} & 1 & \\ 0 & 0 & 0 \end{matrix} \quad \text{(3 by symmetry)}$$

$$\begin{matrix} & 1 & \\ 0 & 2 & 0 \end{matrix}, \begin{matrix} & 0 & \\ 1 & 1 & 1 \end{matrix} \quad \text{(3 by symmetry)}$$

The are also 6 extremal rays which correspond to the Schur sequences

$$\begin{matrix} & 1 & \\ 0 & 2 & 0 \end{matrix}, \begin{matrix} & 0 & \\ 1 & 1 & 0 \end{matrix}, \begin{matrix} & 0 & \\ 0 & 0 & 1 \end{matrix} \quad \text{(6 by symmetry).}$$

The cone $\Sigma(Q,\alpha)$ is a cone over a hexagon.



7. The set of σ-stable dimension vectors

In the previous section we studied, how the σ-stable decomposition of α varies, when σ varies and α is fixed. This led to the description of the faces of $\Sigma(Q,\alpha)$. In this section, we will study how the σ-stable decomposition looks like for a fixed weight σ. Let us define $\overline{\Sigma}(Q,\sigma)$ as the set of all σ-semistable dimension vectors. Notice that

$$\alpha \in \overline{\Sigma}(Q,\sigma) \quad \Leftrightarrow \quad \sigma \in \Sigma(Q,\alpha).$$

Moreover, if we write $\tau = \langle \alpha, \cdot \rangle$ and $\sigma = -\langle \cdot, \beta \rangle$, then

$$\alpha \in \overline{\Sigma}(Q,\sigma) \quad \Leftrightarrow \quad \tau \in \Sigma(Q,\beta),$$

so as a cone, $\overline{\Sigma}(Q,\sigma)$ is the same as $\Sigma(Q,\beta)$ after a linear transformation.

Lemma 33. *If the σ-stable decomposition of α is*

$$\alpha = \alpha_1 \dotplus \alpha_2 \dotplus \cdots \dotplus \alpha_s$$

then the σ-stable decomposition of $m\alpha$ is

$$m\alpha = (m\alpha_1) \dotplus (m\alpha_2) \dotplus \cdots \dotplus (m\alpha_s)$$

where $m\beta$ is equal to $\beta \dotplus \beta \dotplus \cdots \dotplus \beta$ (m copies) if β is real or isotropic, or equal to $m\beta$ otherwise.

Definition 34. For a sequence of dimension vectors $\underline{\alpha} = (\alpha_1, \dots, \alpha_s)$ (all α_i distinct) with $\langle \alpha_i, \alpha_j \rangle \leq 0$ for $i \neq j$, we define a quiver $Q(\underline{\alpha})$ as follows. The set of vertices of $Q(\underline{\alpha})_0$ is equal to $\{1, 2, \dots, s\}$. For $i \neq j$ there are $-\langle \alpha_i, \alpha_j \rangle$ arrows from i to j. There are $1 - \langle \alpha_i, \alpha_i \rangle$ arrows (loops) from i to i.

Lemma 35. *Suppose that $\alpha \in \overline{\Sigma}(Q,\sigma)$. There exists a sequence $\underline{\delta} = \delta_1, \dots, \delta_s$ such that*

1. *$\alpha = \sum_{i=1}^{s} a_i \delta_i$ for some positive rational numbers $a_1, \dots, a_s$;*
2. *$\delta_1, \dots, \delta_s$ are linearly independent dimension vectors;*
3. *each δ_i lies in some extremal ray of the cone $\overline{\Sigma}(Q,\sigma)$.*

Proof. This is trivial. □

Lemma 36. *Suppose that $\alpha, \beta, \delta_1, \dots, \delta_s$ are σ-stable, and $\beta \hookrightarrow \alpha$.*

1. *If α is a nonnegative integral combination of $\delta_1, \dots, \delta_s$ then so is β.*
2. *If α is a nonnegative rational combination of $\delta_1, \dots, \delta_s$ then so is β.*

Proof. Suppose that $\alpha = \sum_{i=1}^{s} a_i \delta_i$ for some integers $a_1, \dots, a_s$. Let V_i be σ-stable of dimension δ_i. Consider the representation

$$V = V_1^{a_1} \oplus V_2^{a_2} \oplus \cdots \oplus V_s^{a_s}.$$

Now V has a semi-stable subrepresentation of dimension β, so β must be a nonnegative integral combination of $\delta_1, \dots, \delta_s$.

The second statement follows from the fact that for each positive integer m we have

$$\beta \hookrightarrow \alpha \quad \Leftrightarrow \quad m\beta \hookrightarrow m\alpha.$$

□

Theorem 37. *Under the assumptions of Lemma 35 above, α is σ-stable if and only if*

1. *either* $\alpha = \delta_i$ *and* δ_i *is a real Schur root for some* i*,*
2. *or* $\langle \delta_i, \alpha\rangle \leq 0$ *and* $\langle \alpha, \delta_i \rangle \leq 0$ *for all* i*,* $Q(\underline{\delta})$ *is path connected and* α *is indivisible if* α *is isotropic.*

Proof. First we prove that the conditions are necessary. Suppose that $\langle \alpha, \delta_i\rangle > 0$. This is only possible when δ_i is a real Schur root. In that case we have $\hom_Q(\alpha, \delta_i) \neq 0$, so $\alpha = \delta_i$. Similarly, if $\langle \delta_i, \alpha\rangle > 0$ then $\alpha = \delta_i$.

Consider the quiver $Q(\underline{\delta})$. Let S_1 be the set of all k where there is a path from i to k and let $S_2 = S \setminus S_1$. Define

$$\alpha_1 = \sum_{i \in S_1} c_i \alpha_i, \qquad \alpha_2 = \sum_{i \in S_2} c_i \alpha_i.$$

There are no arrows from S_1 to S_2. This show that $\langle \alpha_1, \alpha_2\rangle = 0$. Choose an integer m such that mc_i is a positive integer for all i. We have $m\alpha_1 \hookrightarrow m\alpha$. Since α is σ-stable, $m\alpha_1$ must be proportional to $m\alpha$. This can only happen if $S_2 = \emptyset$. We have shown that the quiver $Q(\underline{\delta})$ is path connected.

If α is isotropic, then α must be indivisible because otherwise it would have a nontrivial canonical decomposition.

Clearly if condition (1) is satisfied, then α is σ-stable. Suppose that (2) is satisfied. Suppose that $\beta \hookrightarrow \alpha$ and β is σ-stable. We will show that $\beta = \alpha$. By Lemma 36, $\beta = \sum_{i=1}^{s} b_i \delta_i$ such that the b_i's are nonnegative rational numbers. Define

$$\operatorname{Supp}(\beta) = \{i \mid b_i \neq 0\}$$

We will first prove that $\operatorname{Supp}(\beta) = \operatorname{Supp}(\alpha) = \{1, 2, \ldots, s\}$. Let $T = \operatorname{Supp}(\alpha) \setminus \operatorname{Supp}(\beta)$ and assume that $T \neq \emptyset$. Define

$$\alpha_1 = \sum_{i \in \operatorname{Supp}(\beta)} a_i \delta_i, \quad \alpha_2 = \sum_{i \in T} a_i \delta_i.$$

Now $\alpha = \alpha_1 + \alpha_2$. Because $\langle \delta_i, \delta_j\rangle \leq 0$ for all $i \neq j$, and because there must be at least one arrow from $\operatorname{Supp}(\beta)$ to T, we get $\langle \beta, \alpha_2\rangle < 0$. Since $\operatorname{ext}_Q(\beta, \alpha - \beta) = 0$, we get $\langle \beta, \alpha - \beta\rangle \geq 0$, so $\langle \beta, \gamma\rangle > 0$ with

$$\gamma = \alpha_1 - \beta = \sum_{\alpha_i \in S_\beta} (a_i - b_i)\delta_i.$$

If

$$\gamma = \gamma_1 \dotplus \gamma_2 \dotplus \cdots \dotplus \gamma_r$$

is the σ-stable decomposition of γ, then $\langle \beta, \gamma_j\rangle > 0$ for some j. This means that $\beta = \gamma_j$ but then

$$\langle \beta, \gamma_j\rangle = \langle \beta, \beta\rangle = \langle \beta, \alpha\rangle - \langle \beta, \alpha - \beta\rangle \leq 0.$$

Constradiction.

This shows that $\operatorname{Supp}(\beta) = \operatorname{Supp}(\alpha)$. Let $\gamma = \alpha - \beta$, and assume that $\gamma \neq 0$. Then we find a σ-stable dimension vector γ' such that $\gamma \twoheadrightarrow \gamma'$ and therefore $\alpha \twoheadrightarrow \gamma'$. By a similar argument as before we obtain $\operatorname{Supp}(\alpha) = \operatorname{Supp}(\gamma') = \operatorname{Supp}(\gamma)$. Write $\gamma = \sum_{i=1}^{s} c_i \delta_i$ with c_i a positive rational number for all i. We have

$$0 = \langle \beta, \gamma\rangle = \sum_{i=1}^{s} c_i \langle \beta, \delta_i\rangle.$$

Since $\langle \beta, \beta\rangle \leq 0$ and β is σ-stable, we have that $\langle \beta, \delta_i\rangle \leq 0$ for all i. So we must have that $\langle \beta, \delta_i\rangle = 0$ for all i.

Let $B = \max\{b_1, \dots, b_s\}$. Let $S_1 = \{i \mid b_i = B\}$ and let $S_2 = \{1, 2, \dots, s\} \setminus S_1$. Suppose $S_2 \neq \emptyset$. There must be an arrow from S_2 to S_1, say $k \to j$ with $j \in S_1$ and $k \in S_2$. Then

$$0 = \langle \beta, \delta_j \rangle \leq b_j \langle \delta_j, \delta_j \rangle + b_k \langle \delta_k, \delta_j \rangle.$$

We know that $b_j > b_k$, $\langle \delta_j, \delta_j \rangle \leq 1$, and $\langle \delta_k, \delta_j \rangle \leq -1$. This leads to a contradiction.

So $S_2 = \emptyset$ and $b_i = B$ for all i. From $\langle \beta, \delta_i \rangle = 0$ follows that for every i there is exactly one arrow with tail i. Since $Q(\underline{\delta})$ is path connected, the quiver $Q(\underline{\delta})$ has to be a cycle. Now it easily follows from $\langle \alpha, \delta_i \rangle \leq 0$ that α must be proportional to β and $\langle \alpha, \alpha \rangle = 0$. Since α is indivisible in that case, $\alpha = \beta$. □

Remark 38. The previous theorem also provides us an inductive way of finding the σ-stable decomposition. If α is σ-semistable, but not σ-stable. There are two cases.

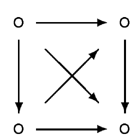
In the first case $\langle \delta_i, \alpha \rangle > 0$ or $\langle \alpha, \delta_i \rangle > 0$ for some extremal dimension vector $\delta_i \in \overline{\Sigma}(Q, \sigma)$. If we know the σ-stable decomposition of the smaller dimension vector $\alpha - \delta_i$, we know the σ-stable decomposition of α.

In the second case $Q(\underline{\delta})$ is not path-connected, say there is no path from i to j. Let S_1 be the set of all k such that there is a path from i to k and let S_2 be the complement. There are no arrows from S_1 to S_2. If we define

$$\alpha_1 = \sum_{i \in S_1} a_i \delta_i, \qquad \alpha_2 = \sum_{i \in S_2} a_i \delta_i.$$

For some m, $m\alpha_1, m\alpha_2$ are dimension vectors, and $m\alpha_1 \hookrightarrow m\alpha$. If we know the σ-stable decomposition of $m\alpha_1$ and $m\alpha_2$ then we know the σ-stable decomposition of $m\alpha$ and α. Now $m\alpha_1$ and $m\alpha_2$ have smaller support than α.

Example 39. Let Q be the quiver



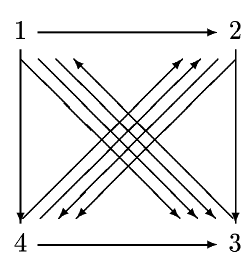
and let σ be the weight

$$\begin{matrix} 1 & -1 \\ 1 & -1 \end{matrix}$$

The extremal rays of the cone $\overline{\Sigma}(Q, \sigma)$ are given by the dimension vectors

$$\delta_1 = \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}, \quad \delta_2 = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \quad \delta_3 = \begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix}, \quad \delta_4 = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}.$$

The quiver $Q(\underline{\delta})$ is

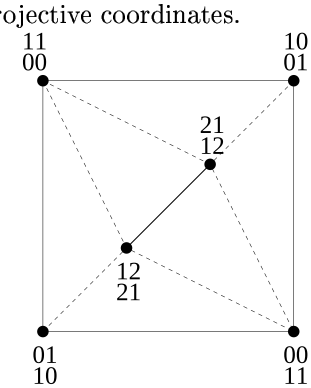


Any σ-stable dimension vector is a nonnegative rational combination of $\delta_1, \delta_2, \delta_4$ or a nonnegative rational combination of $\delta_2, \delta_3, \delta_4$. Suppose α is σ-stable and not equal to $\delta_1, \delta_2, \delta_3, \delta_4$. If α is a nonnegative rational combination $\delta_1, \delta_2, \delta_4$, then because the support has to be path connected, it is actually a nonnegative rational combination of δ_2, δ_4. Similarly, if α is a nonnegative rational combination of $\alpha_2, \alpha_3, \alpha_4$, then it must be in fact a nonnegative rational combination of α_2 and α_4.

Now it easily follows that the set of σ-stable dimension vectors is

$$\delta_1, \quad \delta_2, \quad \delta_3, \quad \delta_4, \quad a\delta_2 + b\delta_4 \ (a, b > 0, a, b \in \mathbb{Z}, a \leq 2b, b \leq 2a).$$

The cone $\overline{\Sigma}(Q, \alpha)$ is a cone over a square. In the diagram below, the coordinates should be interpreted as projective coordinates.



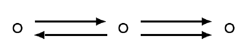
The fat line in the middle of the square corresponds to the imaginary σ-stable dimension vectors. The dashed lines distinguish the regions where the σ-stable decomposition looks different.

Some examples of σ stable decomposition are:

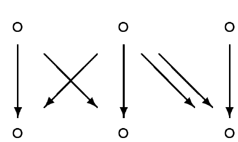
$$\begin{matrix} 5 & 4 \\ 3 & 4 \end{matrix} = \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \dotplus \begin{matrix} 4 & 3 \\ 3 & 4 \end{matrix}$$

$$\begin{matrix} 5 & 4 \\ 3 & 1 \end{matrix} = \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} \dotplus \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \dotplus 2 \cdot \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}.$$

7.1. **σ-stable decomposition for quivers with oriented cycles.** Doubling of the quiver, reduces the σ-stable decomposition for quivers with oriented cycles, to the case of quivers without oriented cycles. Suppose that Q is a quiver with oriented cycles. We define a new quiver $\widehat{Q}$ by $\widehat{Q}_0 = Q_0 \times \{0, 1\}$. For every $a \in Q_1$ we define an arrow $\widehat{a} \in \widehat{Q}_1$ with $t\widehat{a} = (ta, 0)$ and $h\widehat{a} = (ha, 0)$ and for every $x \in Q_0$ we define an arrow $\widehat{x} \in \widehat{Q}_1$ with $t\widehat{x} = (x, 0)$ and $h\widehat{x} = (hx, 0)$. For example, if Q is the quiver



then $\widehat{Q}$ is the quiver



For a Q-dimension vector α, we define a dimension vector $\widehat{\alpha}$ of $\widehat{Q}$ by $\widehat{\alpha}(x, 0) = \widehat{\alpha}(x, 1) = \alpha(x)$ for all $x \in Q_0$. Similarly, if σ is a weight of Q, we define a weight $\widehat{\sigma}$ of $\widehat{Q}$ by $\widehat{\sigma}(x, 0) = \widehat{\sigma}(x, 1) = \sigma(x)$. We define the weight τ of $\widehat{Q}$ by $\tau(x, 0) = 1$ and $\tau(x, 1) = -1$ for all $x \in Q_0$. Note that for any $\alpha \in Q_0$, $\widehat{\alpha}$ is τ-stable.

Proposition 40. *Suppose that α is a dimension vector and σ is a weight for Q. Then α is σ-semistable (stable) if and only if for some large positive integer m, $\widehat{\alpha}$ is $\widehat{\sigma} + m\tau$-semistable (stable).*

Proof. Suppose that α is σ-semistable. If $\gamma \hookrightarrow \widehat{\alpha}$ for some $\widehat{Q}$-dimension vector γ. Note that $\gamma(x,0) \leq \gamma(x,1)$ for all $x \in Q_0$ because $\widehat{\alpha}(x,0) = \widehat{\alpha}(x,1)$, and for a general representation V of dimension $\widehat{\alpha}$ the map $V(\widehat{x}) : V(x,0) \to V(x,1)$ is injective. If $\gamma(x,0) = \gamma(x,1)$ for all $x \in Q_0$ then γ is of the form $\widehat{\beta}$ and $\beta \hookrightarrow \alpha$. Then we have $\widehat{\sigma}(\gamma) = \sigma(\beta) \geq 0$. Also we have $(\widehat{\sigma} + m\tau)(\gamma) = \widehat{\sigma}(\gamma) \leq 0$.

Suppose that $\gamma(x,0) < \gamma(x,1)$ for some $x \in Q_0$. Then $\tau(\gamma) < 0$ so in particular for m large enough we will have $(\widehat{\sigma} + m\tau)(\gamma) < 0$.

Since there are only finitely many subdimension vectors γ, we can choose m large enough such that $(\widehat{\sigma} + m\tau)(\gamma) \leq 0$ for all $\gamma \hookrightarrow \widehat{\alpha}$. This shows that $\widehat{\alpha}$ is $(\widehat{\sigma} + m\tau)$-semistable.

Conversely, assume that $\widehat{\alpha}$ is $\widehat{\sigma} + m\tau$-semistable for some m and $\beta \hookrightarrow \alpha$. Then $\widehat{\beta} \hookrightarrow \widehat{\alpha}$, so $0 \geq (\widehat{\sigma} + m\tau)(\widehat{\beta}) = \widehat{\sigma}(\widehat{\beta}) = \sigma(\beta)$. This shows that α is σ-stable.

A similar statement with stable instead of semistable is easy to prove. ☐

Suppose now that Q is quiver, possible with oriented cycles. Let us consider the 0-stable decomposition. Clearly, every representation of Q is 0-semistable. A representation V is 0-stable if the only subrepresentations are 0 and V itself. In other words, 0-stable representations are exactly simple representations. Notice that if there exists an α-dimensional simple representation, then the general representation of dimension α is simple. Such dimension vectors we will call simple.

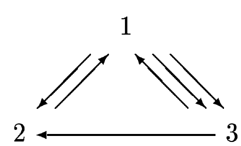
Corollary 41. *Suppose that Q is an arbitrary quiver. For each $x \in Q_0$ we define a dimension vector δ_x by $\delta_x(y) = 0$ for $y \neq x$ and $\delta_x(x) = 1$. A dimension vector α is simple if*

1. *either $\alpha = \delta_x$ and δ_x is real (i.e., $\langle \delta_x, \delta_x \rangle = 1$);*
2. *or $\langle \delta_x, \alpha \rangle \leq 0$ and $\langle \alpha, \delta_x \rangle \leq 0$ for all $x \in Q_0$, the full subquiver of Q with vertices*

$$\mathrm{Supp}(\alpha) := \{x \in Q_0 \mid \alpha(x) \neq 0\}$$

is path connected, and if α is isotropic, then α is indivisible.

Example 42. Consider the quiver



Suppose α that is the dimension vector (a_1, a_2, a_3). We will find necessary and sufficient conditions for α to be a simple dimension vector. Of course α can be equal to δ_1, δ_2, δ_3. The conditions $\langle \delta_i, \alpha \rangle \leq 0$ and $\langle \alpha, \delta_i \rangle \leq 0$ give the inequalities

$$a_1 \leq a_2 + a_3, \; a_2 \leq a_1, \; a_3 \leq a_1 + a_2$$

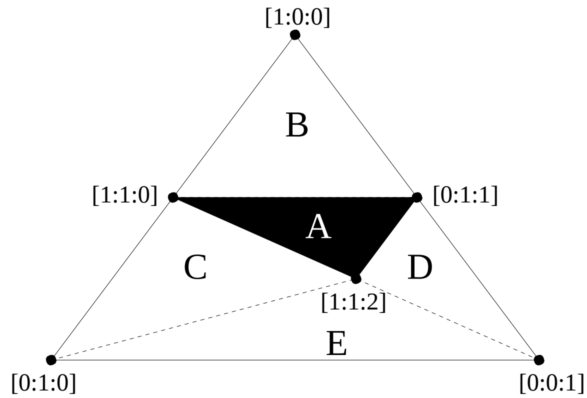
(other inequalities turn out to be redundant). If $a_3 = 0$ and $a_1 = a_2$, then α is isotropic, so we must have that $a_1 = a_2 = 1$ in that case. The only support of α which is not possible (because it is not path connected) is $\{2, 3\}$, but this is already excluded by the inequalities.

From the inequalities and the remarks above it is easy to deduce that set of simple dimension vectors is given by

$$(1,0,0), \; (0,1,0), \; (0,0,1), \; (1,0,0), \; (1,1,0), \text{ and all}$$

$$\{(a_1, a_2, a_3) \in \mathbb{Z}^3 \mid a_1 \leq a_2 + a_3, \; a_2 \leq a_1, \; a_3 \leq a_1 + a_2, a_3 > 0\}$$

In the picture shows how the simple decomposition looks like. We use projective coordinates.



Region A is defined by $a_1 \leq a_2 + a_3,\ a_2 \leq a_1,\ a_3 \leq a_1 + a_2$. This will always define a simple dimension vector except when $a_3 = 0$ (and $a_1 = a_2$). In that case, the simple decomposition is

$$(a, a, 0) = a \cdot (1, 1, 0).$$

Region B is defined by $a_2 + a_3 \leq a_1$. The simple decomposition in this region is

$$(a_1, a_2, a_3) = (a_1 - a_2 - a_3) \cdot (1, 0, 0) \dotplus (a_2 + a_3, a_2, a_3) \qquad \text{if } c_3 > 0 \text{ and}$$

$$(a_1, a_2, 0) = (a_1 - a_2) \cdot (1, 0, 0) \dotplus a_2 \cdot (1, 1, 0).$$

Region C is defined by $a_2 \geq a_1,\ 2a_1 \geq a_3$. The simple decomposition is

$$(a_1, a_2, a_3) = (a_2 - a_1) \cdot (0, 1, 0) \dotplus (a_1, a_1, a_3) \qquad \text{if } c_3 > 0 \text{ and}$$

$$(a_1, a_2, 0) = (a_2 - a_1) \cdot (0, 1, 0) \dotplus a_1 \cdot (1, 1, 0).$$

Region D is defined by $a_1 \geq a_2$ and $a_3 \geq a_1 + a_2$. The simple decomposition here is

$$(a_1, a_2, a_3) = (a_3 - a_1 - a_2) \cdot (0, 0, 1) \dotplus (a_1, a_2, a_1 + a_2).$$

Region E is defined by $a_2 \geq a_1,\ a_3 \geq 2a_1$. The simple decomposition in this region is

$$(a_1, a_2, a_3) = (a_2 - a_1) \cdot (1, 0, 0) \dotplus (a_3 - 2a_1) \cdot (0, 1, 0) \dotplus (a_1, a_1, 2a_1).$$

Example 43. Let Q be the quiver with 3 vertices (labeled 1, 2 and 3), with a loop at each vertex and with arrows $1 \to 2$, $2 \to 3$ and $3 \to 1$. The set of simple dimension vectors is

$$(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1),\ \text{ and all } (a, b, c) \text{ with } a, b, c > 0.$$

Notice that for example a dimension vector of the form $(a, b, 0)$ $(a, b > 0)$ is not simple because its support is not path connected.

8. Littlewood-Richardson coefficients

8.1. **The Klyachko cone.** Irreducible representations of GL_n are parametrized by nonincreasing integer sequences of length n. If λ is such a sequence, then we denote the corresponding representation by V_λ. The Littlewood-Richardson coefficient $c_{\lambda,\mu,\nu}$ is defined by

$$c_{\lambda,\mu,\nu} = \dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{GL}_n}.$$

We would like to study the Klyachko cone

$$\mathcal{K}_n = \{(\lambda, \mu, \nu) \in (\mathbb{Z}^n)^3 \mid \lambda, \mu, \nu \text{ are nonincreasing and } c_{\lambda,\mu,\nu} \neq 0\}.$$

Note that if $c_{\lambda,\mu,\nu} \neq 0$, then $|\lambda| + |\mu| + |\nu| = 0$. The Klyachko cone has dimension $3n - 1$. Let $T_{p,q,r}$ be the quiver with $p + q + r - 2$ vertices:

$$\begin{array}{ccccccccc}
x_1 & \longrightarrow & x_2 & \cdots & x_{p-2} & \longrightarrow & x_{p-1} & & \\
 & & & & & & & \searrow & \\
y_1 & \longrightarrow & y_2 & \cdots & y_{q-2} & \longrightarrow & y_{q-1} & \longrightarrow & x_p \\
 & & & & & & & \nearrow & \\
z_1 & \longrightarrow & z_2 & \cdots & z_{r-2} & \longrightarrow & z_{r-1} & &
\end{array}$$

We use the convention $y_q = z_r = x_p$. In [2] we have seen that if we take the dimension vector

$$\beta = \begin{matrix} 1 & 2 & \cdots & n-1 & \\ 1 & 2 & \cdots & n-1 & n, \\ 1 & 2 & \cdots & n-1 & \end{matrix}$$

for $T_{n,n,n}$, then we can view $\dim \mathrm{SI}(Q, \beta)_\sigma$ as a Littlewood-Richardson coefficient as follows. If σ is given by

$$\sigma = \begin{matrix} a_1 & a_2 & \cdots & a_{n-1} & \\ b_1 & b_2 & \cdots & b_{n-1} & a_n + b_n + c_n, \\ c_1 & c_2 & \cdots & c_{n-1} & \end{matrix}$$

then

$$\dim \mathrm{SI}(Q, \beta)_\sigma = c_{\lambda,\mu,\nu}$$

where

$$\lambda = (a_1 + \cdots + a_n, a_2 + \cdots + a_n, \cdots, a_{n-1} + a_n, a_n),$$
$$\mu = (b_1 + \cdots + b_n, b_2 + \cdots + b_n, \cdots, b_{n-1} + b_n, b_n)),$$
$$\nu = (c_1 + \cdots + c_n, c_2 + \cdots + c_n, \cdots, c_{n-1} + c_n, c_n))$$

We will describe this also in terms of α and β where $\sigma = \langle \alpha, \cdot \rangle$. We will also generalize the above to arbitrary $T_{p,q,r}$ and arbitrary dimension vectors α and β. Let us define

$$\widetilde{c}_{\lambda,\mu,\nu} = \dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{SL}_n}.$$

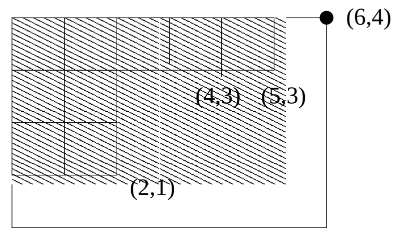
So if $|\lambda| + |\mu| + |\nu| = 0$, then $\widetilde{c}_{\lambda,\mu,\nu} = c_{\lambda,\mu,\nu}$. If $\widetilde{c}_{\lambda,\mu,\nu} \neq 0$, then $|\lambda| + |\mu| + |\nu|$ must be a multiple of n, say mn. In that case $\widetilde{c}_{\lambda,\mu,\nu} = c_{\lambda,\mu,\nu-m\delta}$ where $\delta = (1, 1, \ldots, 1)$.

Definition 44. Let $\underline{x} = (x_1, x_2, \ldots, x_n)$, $\underline{y} = (y_1, y_2, \ldots, y_n)$ be two nondecreasing sequences of nonnegative integers. We define a partition $P(\underline{x}, \underline{y})$ by

$$P(\underline{x}, \underline{y}) = (x_{n-1}^{y_n - y_{n-1}}, x_{n-2}^{y_{n-1} - y_{n-2}}, \ldots, x_1^{y_2 - y_1}, 0^{y_1}).$$

Graphically, this partition can be found as follows. In the Euclidian plane we draw a square with vertices $(0,0)$, $(x_n,0)$, $(0,y_n)$, (x_n,y_n). and we plot the points $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$. We take the region in the square above and left to those points. Viewed in the unit grid, the corresponding partition $P(\underline{x},\underline{y})$ can be read off.

Example 45. If $\underline{x}=(2,4,5,6)$ and $\underline{y}=(1,3,3,4)$ then $P(\underline{x},\underline{y})=(5,2,2,0)$ by the diagram below:.



We consider the quiver $T_{p,q,r}$. Let α and β be dimension vectors. We write $\alpha(x)=(\alpha(x_1),\ldots,\alpha(x_p))$, $\alpha(y)=(\alpha(y_1),\ldots,\alpha(y_q))$ and in a similar way we define $\alpha(z),\beta(x),\beta(y),\beta(z)$.

Lemma 46.

$$\alpha\circ\beta=\widetilde{c}_{\lambda,\mu,\nu}$$

where

$$\lambda=P(\alpha(x),\beta(x)),\ \ \mu=P(\alpha(y),\beta(y)),\ \ \nu=P(\alpha(z),\beta(z)).$$

Proof. This is an easy computation, see [2]. □

Remark 47. Note that if $1/p+1/q+1/r>1$, then $T_{p,q,r}$ is a quiver of finite type and for all dimension vectors α,β we have $\alpha\circ\beta=0$ or $\alpha\circ\beta=1$. For a partition $\lambda=(\lambda_1,\ldots,\lambda_n)$ we define

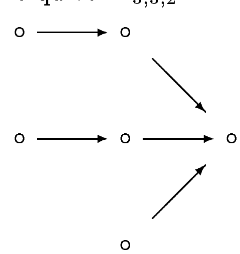
$$j(\lambda)=\#\{i\mid\lambda_{i+1}\neq\lambda_i,1\leq i\leq n-1\}$$

If λ is trivial then $j(\lambda)=0$, if λ is a box then $j(\lambda)=1$ and if λ is L-shaped (fat hook) then $j(\lambda)=2$. A coefficient $\widetilde{c}_{\lambda,\mu,\nu}$ can be obtained from the quiver $T_{p,q,r}$ with $p=j(\lambda)+1,q=j(\mu)+1$ and $r=j(\nu)+1$. In particular, if

$$\frac{1}{j(\lambda)+1}+\frac{1}{j(\mu)+1}+\frac{1}{j(\nu)+1}>1$$

then $\widetilde{c}_{\lambda,\mu,\nu}$ equals 0 or 1.

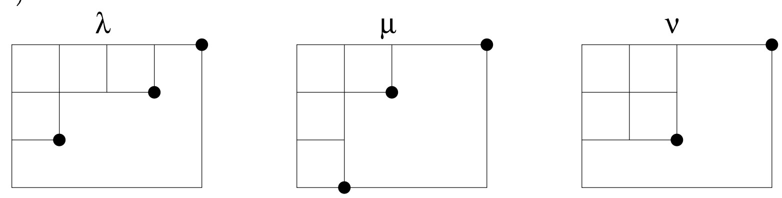
Example 48. Let us consider the quiver $T_{3,3,2}$:



Let α, β be the dimension vectors

$$\begin{array}{cccc} & 1 & 3 & \\ \alpha = & 1 & 2 & 4, \\ & & 2 & \end{array} \qquad \begin{array}{cccc} & 1 & 2 & \\ \beta = & 0 & 2 & 3. \\ & & 1 & \end{array}$$

Now $\alpha \circ \beta$ is equal to the LR-coefficient $\widetilde{c}_{\lambda,\mu,\nu} = 1$ were $\lambda = (3,1)$, $\mu = (2,1,1)$ and $\nu = (2,2)$.



(3,1) (2,1,1) (2,2)

8.2. **Walls of the Klyachko cone.** Let us consider the quiver $Q = T_{n,n,n}$ and the dimension vector

$$\begin{array}{cccccc} & 1 & 2 & \cdots & n-1 & \\ \beta = & 1 & 2 & \cdots & n-1 & n. \\ & 1 & 2 & \cdots & n-1 & \end{array}$$

Lemma 49. *The dimension vector β above is a Schur root.*

We will study the cone $\Sigma(Q,\beta)$ which is essentially the Klaychko cone.

Proposition 50. *For every pair (β_1,β_2) with $\beta = \beta_1 + \beta_2$, β_1, β_2 nondecreasing along arms, $p\beta_1 \circ q\beta_2 = 1$ for all positive integers p and q, the inequality $\sigma(\beta_1) \le 0$ defines a wall of $\Sigma(Q,\beta)$. All nontrivial walls can be uniquely obtained this way.*

Proof. Clearly β_1 and β_2 have at most jumps 1. It follows from Lemma 49 that β_1, β_2 are Schurroots. Now either $\operatorname{ext}_Q(\beta_2,\beta_1) = 0$ or $\hom_Q(\beta_2,\beta_1) = 0$, but the first would give a nontrivial decomposition of β, therefore $\hom_Q(\beta_Q,\beta_1) = 0$ and β_1, β_2 is a quiver Schur sequence. This shows that $\sigma(\beta_1) \le 0$ defines a wall.

For every wall, there exists a Schur sequence (β_1,β_2) such that that $\beta = c_1\beta_1 + c_2\beta_2$ with c_i positive integers and the wall is defined by $\sigma(\beta_1) \le 0$. Note that for $T_{n,n,n}$ a Schur root either has support on one arm (in which case it corresponds to a positive root of A_{n-1}), or it is nondecreasing along each arm. Because β_1 is a subdimension vector of β, it is easy to see that β_1 must also be nondecreasing, but β_2 could have support on one arm. In that case it follows from $\langle \beta_1, \beta_2 \rangle = 0$ that β_2 is simple. The inequalities following from such a quiver sequence are trivial, they say that the partitions λ, μ and ν must be decreasing. If β_1 and β_2 are both nondecreasing along the arms, then it is easy to see that $c_1 = c_2 = 1$. $\square$

Knutson, Tao and Woodward recently proved a similar result. Their result can be translated to the same statement, but with the condition $\beta_1 \circ \beta_2 = 1$ instead of $p\beta_1 \circ q\beta_2 = 1$ for all positive integers p and q. In particular for all dimension vectors α, β for $T_{n,n,n}$ we have

$$\alpha \circ \beta = 1 \quad \Rightarrow \quad p\alpha \circ q\beta = 1 \text{ for all positive integers } p, q.$$

In terms of Littewood-Richardson coefficients the results of Knutson, Tao and Woodward imply that

$$\widetilde{c}_{\lambda,\mu,\nu} = 1 \quad \Rightarrow \quad \widetilde{c}_{N\lambda,N\mu,N\nu} = 1 \text{ for all positive integers } N.$$

This was conjectured before by Fulton.

Example 51. Consider the quiver $T_{3,3,3}$ and

$$\beta = \begin{matrix} 1 & 2 & \\ 1 & 2 & 3. \\ 1 & 2 & \end{matrix}$$

The LR-coefficient $c_{\lambda,\mu,\nu}$ corresponds to $\dim \mathrm{SI}(Q,\beta)_\sigma$ where σ is given by

$$\sigma = \begin{matrix} \lambda_1-\lambda_2 & \lambda_2-\lambda_3 & \\ \mu_1-\mu_2 & \mu_2-\mu_3 & \lambda_3+\mu_3+\nu_3. \\ \nu_1-\nu_2 & \nu_2-\nu_3 & \end{matrix}$$

For example the sequence

$$\begin{matrix} 1 & 2 & & 0 & 0 & \\ 0 & 1 & 2, & 1 & 1 & 1 \\ 0 & 1 & & 1 & 1 & \end{matrix}$$

is a Schur sequence, because $\widetilde{c}^{\,1}_{2,0,0} = 1$. This Schur sequence corresponds to the wall

$$\lambda_1 + \lambda_2 + \mu_2 + \mu_3 + \nu_2 + \nu_3 \leq 0$$

By permuting the arms, we obtain similar inequalities by permuting λ, μ, ν. Other walls are given by the Schur sequences

$$\begin{matrix} 1 & 1 & & 0 & 1 & \\ 1 & 1 & 2, & 0 & 1 & 1 \\ 0 & 1 & & 1 & 1 & \end{matrix} (\widetilde{c}^{\,1}_{1,1,0} = 1), \ \lambda_1 + \lambda_3 + \mu_1 + \mu_3 + \nu_2 + \nu_3 \leq 0, \quad (3 \text{ by symmetry})$$

$$\begin{matrix} 1 & 1 & & 0 & 1 & \\ 0 & 0 & 1, & 1 & 2 & 2 \\ 0 & 0 & & 1 & 2 & \end{matrix} (\widetilde{c}^{\,2}_{11,0,0} = 1), \ \lambda_1 + \mu_3 + \nu_3 \leq 0, \quad (3 \text{ by symmetry})$$

$$\begin{matrix} 0 & 1 & & 1 & 1 & \\ 0 & 1 & 1, & 1 & 1 & 2 \\ 0 & 0 & & 1 & 2 & \end{matrix} (\widetilde{c}^{\,2}_{1,1,0} = 1), \ \lambda_1 + \lambda_3 + \mu_1 + \mu_3 + \nu_1 + \nu_2 \leq 0, \quad (3 \text{ by symmetry}).$$

Besides these, there are 6 trivial walls corresponding to the inequalities $\lambda_1 \geq \lambda_2 \geq \lambda_3$, $\mu_1 \geq \mu_2 \geq \mu_3$ and $\nu_1 \geq \nu_2 \geq \nu_3$. These are given by the Schur sequences

$$\begin{matrix} 0 & 2 & & 1 & 0 & \\ 1 & 2 & 3, & 0 & 0 & 0, \\ 1 & 2 & & 0 & 0 & \end{matrix} \ \lambda_1 \geq \lambda_2, \quad (3 \text{ by symmetry})$$

$$\begin{matrix} 1 & 1 & & 0 & 1 & \\ 1 & 2 & 3, & 0 & 0 & 0, \\ 1 & 2 & & 0 & 0 & \end{matrix} \ \lambda_2 \geq \lambda_3, \quad (3 \text{ by symmetry}).$$

The Klyachko cone $\mathcal{K}_n$ can be found by induction on n. Let us consider again the quiver $Q = T_{n,n,n}$ and the dimension vector

$$\beta = \begin{matrix} 1 & 2 & \cdots & n-1 & \\ 1 & 2 & \cdots & n-1 & n. \\ 1 & 2 & \cdots & n-1 & \end{matrix}$$

We need to find all β_1, β_2 such that $\beta = \beta_1 + \beta_2$ and $\beta_1 \circ \beta_2 = 1$. In that case $\sigma(\beta_1) \leq 0$ defines a wall of $\Sigma(Q, \beta)$. If we take

$$\sigma = \begin{array}{cccc} \lambda_1 - \lambda_2 & \lambda_2 - \lambda_3 & \cdots & \lambda_{n-1} - \lambda_n \\ \mu_1 - \mu_2 & \mu_2 - \mu_3 & \cdots & \mu_{n-1} - \mu_n & \lambda_3 + \mu_3 + \nu_3. \\ \nu_1 - \nu_2 & \nu_2 - \nu_3 & \cdots & \nu_{n-1} - \nu_n \end{array}$$

then $\sigma(\beta_1) \leq 0$ defines a wall of the Klyachko cone. However, note that if $\beta_1 \circ \beta_2 > 0$ then $\sigma(\beta_1) \leq 0$ still gives a true inequality. The fact that β_1 has jumps of at most 1, gives these inequalities a special form, namely, if we take

$$I = \{i \mid \beta_1(x_{i-1}) = \beta_1(x_i), 1 \leq i \leq n\},$$

$$J = \{i \mid \beta_1(y_{i-1}) = \beta_1(y_i), 1 \leq i \leq n\},$$

$$K = \{i \mid \beta_1(z_{i-1}) = \beta_1(z_i), 1 \leq i \leq n\},$$

(by convention $\beta_1(x_0) = \beta_1(y_0) = \beta_1(z_0) = 0$), then the inequality corresponding to β_1 is

$$\sum_{i \in I} \lambda_i + \sum_{i \in J} \mu_i + \sum_{i \in K} \nu_i \leq 0.$$

Note that $\#I = \#J = \#K = \beta_1(x_n)$. To find all inequalities for $\mathcal{K}_n$, we need to find all β_1, β_2 with $\beta = \beta_1 + \beta_2$ and $\beta_1 \circ \beta_2 > 0$. This again is given by a Littlewood-Richardson coefficient for $\mathrm{SL}_{\beta_2(x_n)}$. Since $\beta_2(x_n) < n$ we know necessary and sufficient inequalities for the corresponding LR-coefficient to be nonzero.

8.3. Faces of the Klyachko cone of arbitrary codimension.

Corollary 52. *There is a 1–1 correspondence between Schur sequences $(\beta_1, \ldots, \beta_{l+1})$ such that $\beta = c_1\beta_1 + \cdots + c_{l+1}\beta_{l+1}$ for some positive integers $c_1, \ldots, c_{l+1}$, $\beta_1, \ldots, \beta_{l+1}$ are Schur roots, and $\beta_i \circ \beta_j = 1$ for all $i < j$.*

However, If $l > 1$ then the c_i may be larger than 1, and it is also not so easy to decide whether β_i is a Schur root. This makes it more difficult to find the faces of higher codimension.

Corollary 53. *Suppose that (λ, μ, ν) lies in a face F of $\mathcal{K}_n$ of codimension l. Let $j(\lambda), j(\mu), j(\nu)$ be the number of jumps in λ, μ, ν respectively.*

1.
$$j(\lambda) + j(\mu) + j(\nu) \leq 4n - 4 - l,$$
2. *if $c_{\lambda,\mu,\nu} > 1$ then*
$$j(\lambda) + j(\mu) + j(\nu) \leq 4n - 6 - l.$$

The cone $\Sigma(Q, \beta)$ has one 0-dimensional face, namely $\{0\}$. This corresponds to the the 2-dimensional face of $\mathcal{K}_n$ consisting of all (λ, μ, ν), $\lambda = (a, \ldots, a)$, $\mu = (b, \ldots, b)$, $\nu = (c, \ldots, c)$ with $a + b + c = 0$. Interesting is also to study the extremal rays of the cone $\Sigma(Q, \beta)$ (or equivalently the 3-dimensional faces of $mathcal K_n$). They span the cone $\Sigma(Q, \beta)$ (or the Klyachko cone). The codimension of the extremal rays is $3n - 4$. One interesting question is, whether $c_{\lambda,\mu,\nu}$ whenever (λ, μ, ν) is on an extremal ray of $\mathcal{K}_n$. We first give a positive result in this direction:

Corollary 54. *If (λ, μ, ν) is in an extremal ray of $\mathcal{K}_n$, and $n \leq 7$, then $c_{\lambda,\mu,\nu} = 1$.*

Proof. Suppose that $c_{\lambda,\mu,\nu} > 1$. Then $j(\lambda) + j(\mu) + j(\nu) \geq 6$. So

$$6 \leq 4n - 6 - (3n - 4)$$

and we deduce that $n \geq 8$. Contradiction. □

Example 55. We study $T_{8,8,8}$ with the weight

$$\sigma = \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & \end{array}$$

The σ-stable decomposition of

$$\beta = \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \end{array}$$

is

$$\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 2 & \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & \end{array} \dotplus \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{array} \dotplus$$

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & \end{array} \dotplus \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \end{array} \dotplus$$

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \end{array} \dotplus \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \end{array} \dotplus$$

$$\delta_{x_2} \dotplus 2 \cdot \delta_{x_3} \dotplus 3 \cdot \delta_{x_4} + \delta_{x_6} + 2 \cdot \delta_{x_7} \dotplus$$
$$\delta_{y_1} \dotplus 2 \cdot \delta_{y_2} \dotplus \delta_{y_4} \dotplus 2 \cdot \delta_{y_5} \dotplus \delta_{y_7} \dotplus$$
$$\delta_{z_1} \dotplus 2 \cdot \delta_{z_2} \dotplus \delta_{z_4} \dotplus 2 \cdot \delta_{z_5} \dotplus \delta_{z_7}.$$

In the σ-stable decomposition of β, there are 21 distinct Schur roots. The quiver $T_{8,8,8}$ has 22 vertices, so this proves that σ is in an extremal ray of $\Sigma(Q,\beta)$. The LR-Richardson coefficient corresponding to β and σ is $\widetilde{c}_{\lambda,\mu,\nu}$ where

$$\lambda = (2,1,1,1,1,0,0,0),\ \mu = (2,2,2,1,1,1,0,0),\ \nu = (2,2,2,1,1,1,0,0).$$

The value of $\widetilde{c}_{\lambda,\mu,\nu}$ is 2. In fact, for any N we have $\widetilde{c}_{N\lambda,N\mu,N\nu} = N + 1$.

8.4. **A multiplicative formula for Littlewood-Richardson coefficients.** Let β and $T_{n,n,n}$ as before.

Proposition 56. *Suppose $\beta = \beta_1 + \beta_2$ and $\beta_1 \circ \beta_2 = 1$. The inequality translates to*

$$\sum_{i \in I} \lambda_i + \sum_{i \in J} \mu_i + \sum_{i \in K} \nu_i \leq 0.$$

where I, J, K are subsets of $S = \{1, 2, \ldots, n\}$ of the same cardinality. Suppose that

$$\lambda = (\lambda_1, \ldots, \lambda_n), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$
$$\mu = (\mu_1, \ldots, \mu_n), \quad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n,$$
$$\nu = (\nu_1, \ldots, \nu_n), \quad \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n,$$
$$\lambda^* = (\lambda_{i_1}, \cdots, \lambda_{i_r}), \quad I = \{i_1, i_2, \ldots, i_r\},$$
$$\lambda^\# = (\lambda_{\tilde{\imath}_1}, \cdots, \lambda_{\tilde{\imath}_{n-r}}), \quad S \setminus I = \{\tilde{\imath}_1, \tilde{\imath}_2, \ldots, \tilde{\imath}_{n-r}\},$$
$$\mu^* = (\mu_{j_1}, \cdots, \mu_{j_r}), \quad J = \{j_1, j_2, \ldots, j_r\},$$

$$\mu^{\#} = (\mu_{\tilde{j}_1}, \cdots, \mu_{\tilde{j}_{n-r}}), \quad S \setminus J = \{\tilde{j}_1, \tilde{j}_2, \ldots, \tilde{j}_{n-r}\},$$
$$\nu^* = (\nu_{k_1}, \cdots, \nu_{j_r}), \quad K = \{k_1, k_2, \ldots, k_r\},$$
$$\nu^{\#} = (\nu_{\tilde{k}_1}, \cdots, \nu_{\tilde{k}_{n-r}}), \quad S \setminus K = \{\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_{n-r}\},$$

then we have

$$c_{\lambda,\mu,\nu} = c_{\lambda^*,\mu^*,\nu^*} c_{\lambda^{\#},\mu^{\#},\nu^{\#}}.$$

Example 57. For $n = 8$, $\beta = \beta_1 + \beta_2$ with

$$\beta_1 = \begin{matrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & \\ 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5, \\ 0 & 0 & 1 & 2 & 3 & 3 & 4 & \end{matrix} \quad \beta_2 = \begin{matrix} 0 & 1 & 1 & 2 & 2 & 3 & 3 & \\ 0 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 2 & 2 & 2 & 3 & 3 & \end{matrix}$$

Now $\beta_1 \circ \beta_2 = \tilde{c}_{321,321,300} = 1$, so β_1, β_2 is a Schur sequence. The corresponding inequality for the Klyachko cone is

$$\lambda_1 + \lambda_3 + \lambda_5 + \lambda_7 + \lambda_8 + \mu_1 + \mu_3 + \mu_5 + \mu_7 + \mu_8 + \nu_3 + \nu_4 + \nu_5 + \nu_7 + \nu_8 \leq 0$$

or equivalently

$$\lambda_2 + \lambda_4 + \lambda_6 + \mu_2 + \mu_4 + \mu_6 + \nu_1 + \nu_2 + \nu_6 \geq 0.$$

If now these inequalities are equalities, then

$$c_{\lambda,\mu,\nu} = c_{\lambda^*,\mu^*,\nu^*} c_{\lambda^{\#},\mu^{\#},\nu^{\#}}$$

where

$$\lambda^* = (\lambda_1, \lambda_3, \lambda_5, \lambda_7, \lambda_8), \ \mu^* = (\mu_1, \mu_3, \mu_5, \mu_7, \mu_8), \ \nu^* = (\nu_3, \nu_4, \nu_5, \nu_7, \nu_8)$$

and

$$\lambda^{\#} = (\lambda_2, \lambda_4, \lambda_6), \ \mu^{\#} = (\mu_2, \mu_4, \mu_6), \ \nu^{\#} = (\nu_1, \nu_2, \nu_6).$$

For example, take $\lambda = \mu = (8, 4, 4, 2, 2, 0, 0, 0)$, $\nu = (-2, -3, -3, -3, -4, -7, -8, -10)$. Then $c_{\lambda,\mu,\nu} = c_{\lambda^*,\mu^*,\nu^*} c_{\lambda^{\#},\mu^{\#},\nu^{\#}}$ where $\lambda^* = \mu^* = (8, 4, 2, 0, 0)$, $\nu^* = (-3, -3, -4, -8, -10)$, $\lambda^{\#} = \mu^{\#} = (4, 2, 0)$, $\nu^* = (-2, -3, -7)$. Indeed, $c_{\lambda^*,\mu^*,\nu^*} = 5$, $c_{\lambda^{\#},\mu^{\#},\nu^{\#}} = 2$ and $c_{\lambda,\mu,\nu} = 10$.

References

[1] W. Crawley-Boevey, *Exceptional sequences of representations of quivers*, Canadian Math. Soc. Conf. Proceedings, Vol **14** (1993), 117–124.

[2] H. Derksen, J. Weyman, *Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients*, to appear in Journal of the AMS.

[3] H. Derksen, J. Weyman, *On the canonical decomposition of quiver representations*, preprint.

[4] A.D. King, *Moduli of representations of finite dimensional algebras*, Quart. J. Math. Oxford **(2) 45** (1994), 515–530.

[5] V. Kac, *Infinite root systems, representations of graphs and Invariant Theory*, Invent. Math. **56** (1980), 57–92.

[6] V. Kac, *Infinite Root Systems, Representations of Graphs and Invariant Theory II*, J. Algebra **78** (1982), 141–162.

[7] A. Rudakov, *Stability for an abelian category*, J. Algebra **197** (1997), 231–245.

[8] A. Schofield, *Semi-invariants of Quivers*, J. London Math. Soc. **43** (1991), 383–395.

[9] A. Schofield, *General Representations of Quivers*, Proc. London Math. Soc. (3) **65** (1992), 46–64.