We construct a one-dimensional ring \( R \) whose local rings are all fields and discrete Noetherian valuation domains such that \( \text{Spec} ( R ) \) is connected but \( R \) is not a domain.

Let \( H \) be an infinite totally ordered set with the property that between any two distinct elements of \( H \) there is another element. Let \( \{ x_h : h \in H \} \) be a family of indeterminates indexed by \( H \); order them so that \( x_h < x_k \) precisely when \( h < k \). We form a commutative multiplicative semigroup \( S \) whose elements consist of 1 and the positive powers of the individual \( x_h \). The multiplication is given by the rule that \( x_h^m x_k^n \) is

1. \( x_h^m \) if \( h < k \)
2. \( x_h^m x_k^n \) if \( h = k \)
3. \( x_k^n \) if \( k < h \) (forced from (1) by commutativity).

The operation is easily checked to be commutative and associative: no matter how one inserts parentheses, \( x_g^n x_h^m x_k^l \) will be the least of the three variables occurring, with the exponent that is the sum of the exponents with which it occurs in the product. It is easy to verify this by considering the three cases determined by the number of occurrences of the smallest variable among the three terms.

Now let \( K \) be a field, and let \( R \) be the semigroup ring of \( S \) over \( K \). We shall show that \( R \) has connected spectrum and is locally a domain but not a domain. We first show that \( R \) contains no idempotents except 0, 1 (thus, \( \text{Spec} ( R ) \) is connected) and no nilpotent except 0. Suppose one had such an element \( r \). It cannot be a constant. Consider the highest degree terms that occur, and from among them pick the one with the largest variable: suppose that this term is \( cx_k^d \), where \( c \in K \setminus \{ 0 \} \). Then the expansion of \( r^2 \) involves a term \( c^2 x_k^{2d} \) and only one such term occurs: it cannot be canceled. This term shows that \( r^2 \neq 0 \), and also that \( r^2 \neq r \), since \( \text{deg}(r^2) > d \).

Evidently \( R \) is not a domain, since whenever \( h < k \), \( x_h = x_h x_k \) and \( x_h(1 - x_k) = 0 \). (These relations generate the ideal of relations on the \( x_h \), although we do not need this.) It remains to show that \( R \) is locally a domain, and so we study the prime ideals of \( R \). Call \( J \subseteq H \) an upper (respectively, lower) interval if whenever \( h \in J \) and \( k > h \) (respectively, \( k < h \)) then \( k \in J \) as well.

Given a prime ideal \( P \), let \( J_P \) denote the subset of \( H \) consisting of those \( h \) such that \( x_h \notin P \). Note that if \( x_h \notin P \) then for all \( k > h \), \( x_h(1 - x_k) = 0 \) implies that \( 1 - x_k \in P \), and so \( k \in J_p \) as well. Hence, \( J_P \) is an upper interval in \( H \). Evidently, the set \( I_P \) of \( h \) such that \( x_h \in P \) is a lower interval. Note that \( I_P, J_P \) give a partition of \( H \) such that every element of \( I_P \) is less than every element of \( J_P \). We now consider what happens when we localize at \( P \). If \( k \) is an element of \( J_P \) but not a least element, then we can choose \( x_h \notin P \) with \( h < k \), and the equation \( x_h(1 - x_k) = 0 \) forces \( x_k \) to be identified with 1 in the localization for all \( k \) except possibly the least element in \( J_P \). If \( g \) is any element of \( I_P \) other than the greatest, we can choose \( h \in I_P \) with \( g < h \), and then the equation \( x_g(1 - x_h) = 0 \) forces \( x_g \) to become 0 in the localization. Our assumption on the totally ordered set implies that either \( I_P \) has no greatest element, or \( J_P \) has no least element.

If there is no greatest element in \( I_P \) and no least element in \( J_P \) the localization is \( K \). If \( x_h \) is greatest in \( I_P \) or least in \( J_P \), all other \( x_k \) are identified with 0 or 1 after localizing at \( P \). The local ring one gets is a localization of the polynomial ring \( K[x_h] \). This shows that \( R \) is locally a domain and has Krull dimension 1. \( \square \)