 Problem Set #1: Solutions

1. (a) The ring generated is spanned over K by $\{x^t : t \in T\}$ where T is the set of all integers of the form $mh + nk$ for $m, n \in \mathbb{N}$. Evidently, this set is contained in $d\mathbb{N}$, the nonnegative multiples of d. It suffices to show that T contains dr for all $r \gg 0$. Replacing h and k by h/d and k/d we reduce to the case where $d = 1$. When $\mathrm{GCD}(h, k) = 1$, k is invertible mod h. Hence, the integers $S = \{0, k, 2k, \ldots, (h-1)k\}$ represent all possible residue classes mod h. If $N \geq (h-1)k$, choose $bk \in S$ with the same residue mod h that N has. Then h dividies $N - bk \geq 0$, so that $N = ah + bk$ with $a, b \in \mathbb{N}$ □

(b) We use induction on s. The case $s = 1$ is obvious and $s = 2$ is part (a). Assume $s \geq 3$. Let e be the GCD of $h_1, \ldots, h_{s-1}$. By the induction hypothesis, the ring contains x^{ne} for all $n \geq N$. Choose such an n relatively prime to h_s. Then $\mathrm{GCD}(ne, h_s) = \mathrm{GCD}(e, h_s) = d$, and the result follows from the case where $s = 2$. □

2. (a) Let f, g, h denote $x^{2n-1}, x^{2n} + x, x^{2n+1}$ respectively. Then $g^2 - 2h - fh = x^2$, and so $x = g - (g^2 - 2h - fh)^n$. □

(b) Let $u = x^3 + x$. Since $u^2 \in A = K[x^2]$, $u^{2n} \in A$ for all $n \in \mathbb{N}$, and $u^{2n+1} = u^{2n}u \in Au$. Hence, $T = A[u] = A + Au$. The highest term of an element of $Au - \{0\}$ has odd degree, and so $A \cap Au = 0$, so that $T = A \oplus Au$ and the representation is unique. If an element $a' + au$ of T with $a, a' \in A$ has highest degree $2k + 1$ the highest degree term must come from the highest degree term of au, which cannot be canceled by any term of a', and so is $2r + 3$ where $2r = \deg(a)$. Clearly, $x \notin T$. □

(c) $x = (x^3 + x)/(x^2 + 1)$. □

3. I is proper since $u \neq 0$. To show that I is maximal it suffices to show that if $a \in R - I$ then the image of a has an inverse in R/I, i.e., R/I is a field. Since $a \notin I$, $au \neq 0$. Therefore, $u \in auR$ and we can choose $b \in R$ such that $u = aub$ and $(ab - 1)u = 0$. Hence $ab - 1 \in I$, and b represents an inverse for a in R/I. □

Now suppose $f \in R - I$. If f is a zerodivisor, then $J = \mathrm{Ann}_R(f) \neq 0$. Hence $u \in J$, which means that $fu = 0$. But then $f \in I$, a contradiction. □

4. The ideals of R/I correspond bijectively to the ideals J or R that contain I: under this correspondence, J/I corresponds to J, its inverse image in R. Since $(R/I)/(J/I) \cong R/J$, J is prime (.e., R/J is a domain) if and only J/I is prime. This proves that $\mathrm{Spec}\,(h)$ is a bijection of $\mathrm{Spec}\,(R/I)$ onto $V(I) \subseteq \mathrm{Spec}\,(R)$. It remains to show that the inverse of $\mathrm{Spec}\,(h)$ is continuous. There is a base for the open sets in $V(I)$ consisting of sets of the $U_a = D(a) \cap V(I)$, where $a \in R$ and $D(a) = \mathrm{Spec}\,(R) - V(a)$. Let $\overline{a}$ be the image of a in R/I. For $P \supseteq I$, we have that $a \notin P$ iff $\overline{a} \notin P/I$. Hence, the inverse image of U_a under $(\mathrm{Spec}\,(h))^{-1}$, which is the same as the image of U_a under $\mathrm{Spec}\,(h)$, is $D(\overline{a})$ in $\mathrm{Spec}\,(R/I)$, and is open. □

(b) Let Z be the closure of $\mathcal{P}$: it has the form $V(I)$ and is the smallest closed set containing all primes in $\mathcal{P}$. This means that I is the largest ideal such that $I \subseteq P$ for all $P \in \mathcal{P}$, which is precisely the intersection of the family $\mathcal{P}$. □

5. (a) In $\mathcal{C}(X)$ f is unit if and only if it does not vanish. Given finitely many functions $f_1, \ldots, f_n$ in an ideal I, their zero sets $f_i^{-1}(0)$ must meet: in fact, the intersection is the zero set of $f_1^2 + \cdots + f_n^2$. Note also that the map $\mathcal{C}(X) \to \mathbb{R}$ that sends $f \mapsto f(x)$ is a ring homomorphism, and is surjective: the constant functions already map onto $\mathbb{R}$. It has kernel m_x, and since $\mathcal{C}(R)/m_x \cong \mathbb{R}$, every ideal m_x is maximal. Note that if $x \neq y$, $\{x, y\}$ is a closed set in X that is a discrete space. The function that is 0 on x and 1 on y extends continuously to all of X. Hence, $m_x \neq m_y$.

Let I be any proper ideal. The zero sets of the elements of I are closed sets with the finite intersection property. Since X is compact, they have nonempty intersecton. Let x be a point of the intersection. Then all functions in I vanish at x. Hence, $I \subseteq m_x$. Therefore, if I is maximal, $I = m_x$. $\square$

(b) The sets $V(f) \cap \mathrm{MaxSpec}\big(\mathcal{C}(X)\big)$ are a base for the closed sets. Under the bijection, these correspond to the zero sets $f^{-1}(0)$ in X. Therefore, it suffices to show that every closed set Z in X is an (infinite) intersection of zero sets of functions in $\mathcal{C}(X)$. Let $x \in X - Z$. Then x and Z have disjoint open neighborhoods, and so the function f on the closed set $\{x\} \cup Z$ that is 1 on x and 0 on Z extends to a continuous $\mathbb{R}$-valued function on X. The zero set of f contains Z but not x. This shows that Z is an intersection of zero sets. $\square$

6. Note that $T_M(u)$ is R-linear: $rL + sL'$ maps to $(rL + sL')(u) = rL(u) + sL'(u) = rT_M(u)(L) + sT_M(u)(L')$. Second, $T_M(u)$ is R-linear in u, since $T_M(ru + sv)$ evaluated on L is $L(ru + sv) = rL(u) + sL(v) = rT_M(u)(L) + sT_M(v)(L)$. Finally, we must show that given an R-linear map $f : M \to N$, the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ {\scriptstyle T_M}\downarrow & & \downarrow{\scriptstyle T_N} \\ D\big(D(M)\big) & \xrightarrow[D(D(f))]{} & D\big(D(N)\big) \end{array}$$

commutes. If $u \in M$, $T_N\big(f(u)\big)$ maps $L \in \mathrm{Hom}_R(N, W)$ to to $L\big(f(u)\big)$. On the other hand, since $D\big(D(f)\big)$ maps $g : \mathrm{Hom}_R\big(\mathrm{Hom}_R(M, W),\, W\big)$ to the functional whose value on $L \in \mathrm{Hom}_R(N,\, W)$ is $g(L \circ f)$, we may apply this with $g = T_M(u)$ to obtain that $\big(D\big(D(f)\big) \circ T_M\big)(u)$ maps L to $\big(D\big(D(f)\big)(T_M(u))\big)(L) = T_M(u)(L \circ f) = (L \circ f)(u) = L\big(f(u)\big)$ as well, as required. $\square$