Problem Set #3: Solutions

1. We have that u is represented as c/f^m and v as d/g^n. Since these become equal in R_{fg}, we can choose k such that $(fg)^k(g^n c - f^m d) = 0$. Since f, g generate the unit ideal, so do $F = f^{m+k}$, $G = g^{n+k}$. Let $a = f^k c$ and $b = g^k d$. Then $u = a/F$, $v = b/G$, we have that $Ga = Fb$, and F, G generate the unit ideal, say $yF + zG = 1$. Consider $ay + bz \in R$. In R_f, $(ay+bz)/1 = (ayF+bzF)/F = \big(a(1-zG)+bz\big)/F = \big(a+z(bF-aG)\big)/F = a/F$, and in R_g we have that $(ay+bz)/1 = b/G$ similarly. If $r' \in R$ has the same property, then $(r-r')/1$ is 0 in R_f, and so $r-r'$ is killed by a power of f. Similarly, it is killed by a power of g, and so it is killed by the unit ideal and is 0. $\square$

2. Primes of S lying over P are those that contain PS and are disjoint from $R-P$. These correspond bijectively via contraction to the primes of the fiber $(R-P)^{-1}R[x]/PR[x] \cong (R-P)^{-1}(R/P)[x] \cong \kappa[x]$. Each prime of $\kappa[x]$ is either maximal, in which case it is generated by an irreducible polynomial g, or else (0). The contraction of (0) is $PR[x]$. Given an irreducible polynomial $g \in R[x]$ we may multiply it by a nonzero element of R/P to clear denominators so that its coefficients are in R/P. We may then lift it to an element $f \in R[x]$ as described without changing its degree. $\square$

3. (a) Let $f \in R$ represent an element $\overline{f}$ of R/I not in P/I, a prime of R. Then $f \notin P$, so there is a maximal ideal m of R not containing f that contains P. But then m/I is a maximal ideal of R/I containing P/I but not $\overline{f}$. $\square$

(b) For the first statement, it suffices to show that if $P \in \operatorname{Spec}(R)$ maximal with respect to not containing f it is maximal in R. Since P is an intersection of maximal ideals, we can choose a maximal ideal m of R that contains P but not f. By the maximality of P, $P = m$. $\square$ For the second statement, let $Q \in \operatorname{Spec}(R_f)$ and let P be its contraction to R. Let $g/f^n \in R_f \notin Q$. Then $g \notin P$, and $f \notin P$, so that there is a maximal m of R not containing fg. Then mR_f is a maximal ideal of R_f not containing g/f^n. $\square$

4. (a) Map $R[x_1, \ldots, x_n] \twoheadrightarrow S$. By **3.** (a) it suffices to prove that $R[x_1, \ldots, x_n]$is a Hilbert ring, and by induction on n this reduces to the case $n = 1$. Hence, we may assume $S = R[x]$. Let Q be a prime ideal of $R[x]$ lying over P in R. Let $g \in R[x] - Q$. By **2.** there are two cases. One is that $Q = PR[x]$. Then g has a $c \notin P$. Choose a maximal ideal m of R with $c \notin m$. Thus, the image γ of g in $K[x]$ where $K = R/m$ is nonzero. Choose an irreducible polynomial f in $K[x]$ that does not divide g, e.g., x if $g = 1$ and any irreducible factor of $g-1$ otherwise. Then g is not in the maximal ideal of $R[x]$ that is the kernel of the composite surjection $R[x] \twoheadrightarrow K[x] \twoheadrightarrow K[x]/(f)$ (note that $K[x]/(f)$ is a field). In the second case, Q is the contraction of $\overline{f}\kappa[x]$ as in **2.**. The image of g is in $\kappa[x]/(f) - \{0\}$. Let $h \in \kappa[x]$ represent the inverse of g, so that $hg = 1 \bmod f\kappa[x]$, i.e., $hg = 1 + sf$. Then we can choose $b \in R - P$ such that $h_1 = bh$ and $s_1 = bs_1$ both have coefficients in R/P. We then have that $h_1 g = b + s_1 f$ in $(R/P)[x]$. Let m be any maximal ideal of R that does not contain ab, where a is the leading coefficient of f, and let f_0 be any irreducible factor of the image of f modulo m. Then $mR[x] + f_0R[x]$ is a maximal ideal of $R[x]$ that contains f and not ab, and so it does not contain $h_1 g$. This implies it does not contain g. $\square$

(b) Write $S = R[u_1, \ldots, u_n]$. We show that for every k, $0 \le k \le n$, $R[u_1, \ldots, u_{n-k}]$ is a field. The case $n = 0$ follows from the hypothesis. At the inductive step, we have, with $R' = [u_1, \dots, u_{n-k-1}]$, that $R'[u]$ is a field, where $u = u_{n-k}$, and we want to show that R' is a field. Thus, we have reduced to the case where $n = 1$. If $S = R[u]$ we cannot have that u is transcendental over R, for then u has no inverse in the polynomial ring $R[u]$. Hence, u satisfies some polynomial equation $(*) \quad a_k u^k + \cdots + a_1 u + a_0 = 0$ over R, and with $a = a_k$ we have that au is integral over R: it satisfies the equation $(au)^k + a_1 a(au)^{k-1} + \cdots + a^{k-1}a_0 = 0$ obtained by multiplying $(*)$ by a^{k-1}. Then $R[au]$ cannot be a field, since it is an integral extension of R and R is not a field. Thus, the Hilbert ring $R_1 = R[au]$ is not a field but becomes a field when we invert a, since $R_1[1/a]$ contains u and $R[u]$ is a field. Then a is not a unit of R_1, but R_1 has a maximal ideal $\mathcal{M}$ that does not contain a, since it is a Hilbert ring. $\mathcal{M}$ expands to a nonzero maximal ideal of $R_1[1/a]$, a contradiction. $\square$

(c) Let m be a maximal ideal of S. Let $P = m \cap R$. Then R/P is a Hilbert domain that injects into the field S/m, which is finitely generated as an (R/P)-algebra, and so R/P is a field by part (b), i.e., P is maximal. $\square$

5. (a) (a, b) is in the image of f iff the equations $x = a$, $1 + xy = b$ have as solution, and this is the case iff $1 + ay = b$ has a solution. This is the case if $a \neq 0$, but if $a = 0$ there is a solution iff $b = 1$. Thus, the image of the map is $T = D(x) \cup \{(0, 1)\}$. Since K^2 is irreducible, $D(x)$ is dense, and so T is not closed. It is also not open since $T \cap V(x)$, which is one point, is not open in $V(x)$. (b) Solving $x + y(1 + xy) = a$ and $1 + xy = b$ is equivalent to solving $x + by = a$ and $1 + xy = b$. The first equation yields $x = a - by$ and the second becomes $(*)$ $1 + (a - by)y = b$. In all cases, the value of y uniquely determines x. If $b \neq 0$ $(*)$ becomes $y^2 - (a/b)y + (1 - 1/b) = 0$ which always has a solution because K is algebraically closed. If $\mathrm{char}(K) = 2$, $(*)$ has two solutions unless $a = 0$, when it has one solution. If the $\mathrm{char}(K) \neq 2$, there are two solutions unless $(a/b)^2 - 4(1 - 1/b) = 0$, and then there is a unique solution. If $b = 0$, $(*)$ becomes $1 + ay = 0$, which has a unique solution unless $a = 0$, and then there is no solution. Hence, the image of the map is $K^2 - \{(0, 0)\}$, and the preimage of each point in K^2 contains one or two points as described above. Note that the set of points whose preimage contains only one point is infinite.

6. (a) Let $a_1, \dots, a_n \in K$ be distinct. It suffices to map $P_1, \dots, P_n$ to $(a_1, 0), \dots, (a_n, 0)$ (one may compose this map with the inverse of a map that sends $Q_1, \dots, Q_n$ to these). Use induction on n. If $n = 1$ $(x, y) \mapsto (x + a_1 - a, y - b)$ sends (a, b) to $(a_1, 0)$. For the inductive step, first apply an invertible regular map F that sends $P_1, \dots, P_{n-1}$ to $(a_1, 0), \dots, (a_{n-1}, 0) = S$. Let $F(P_n) = (a, b)$. All we need is an invertible regular map that fixes S and sends (a, b) to $(a_n, 0)$. Let $g = (x - a_1) \cdots (x - a_{n-1})$, and let $c = g(a_n) \neq 0$. If $b \neq 0$, let $m = (a_n - a)/b$ and apply $(x, y) \mapsto (x + my, y)$ followed by $(x, y) \mapsto \big(x, -(b/c)g(x) + y\big)$. Both maps fix S. The first sends (a, b) to (a_n, b) and the second sends $(a_n, \underline{\ })$ to $(a_n, 0)$. If $b = 0$ then $g(a) \neq 0$ and we may apply $(x, y) \mapsto (x, g(x) + y)$ to reach the case where $b \neq 0$. $\square$

(b) The map g in **5.** (b) has image $K^2 - \{(0, 0)\}$. By part (a), we can miss any one chosen point. Use induction on n. Let $P_1, \dots, P_{n-1} \in K^2$ be such that $g^{-1}(P_i) = \{Q_i\}$ has just one point for each i. Let h be a regular map whose image is $K^2 - \{Q_1, \dots, Q_{n-1}\}$. Then $g \circ h$ has image precisely $K^2 - \{P_1, \dots, P_{n-1}, (0, 0))\}$. By part (a) we can compose with an invertible regular map so as to miss any other set of n points. $\square$