1. Let $\mathbb{Z}_+$ be the set of positive integers. Let $T$ be the polynomial ring $K[x, y]$ in two variables over a field $K$ and let $S = K[x, y][x/y^n : n \in \mathbb{Z}_+]$, which is a subring of the fraction field of $T$. The elements $x$, $y$, and $\{x/y^n : n \in \mathbb{Z}_+\}$ generate a maximal ideal $\mathcal{M}$ of $S$. Let $J = x\mathcal{M}$, and let $R = S/J$. Let $u$ be the image of $x$ in $R$. Show that $u$ is a nonzero but that $u$ is in every nonzero ideal of $R$. Show that the annihilator of $u$ in $R$ is $m/J$. Show also that the ring $K[y]$ is a homomorphic image of $R$, so that $R$ is not quasilocal. (Clearly, $y \in m$ is not nilpotent.)

2. Let $R$ be a local ring, and let $f$ be an element of $R$ that is not a zerodivisor. Suppose that $R/fR$ is an integral domain. Prove that $R$ is an integral domain.

3. Let $R$ be a ring and $I$, $J$ ideals of $R$.
   (a) Show that the kernel of the map $R/J \otimes_R I \to R/J$ obtain by applying $R/J \otimes_R -$ to the inclusion $I \subseteq R$ is $(I \cap J)/IJ$.
   (b) Show that if $R/J$ is flat, then $I \cap J = IJ$ for every ideal $I$ of $R$.
   (c) Show conversely that if $I \cap J = IJ$ for every ideal $I$ of $R$, then $R/J$ is flat as an $R$-module.

4. Let $K$ be a field, and let $Y, X_1, X_2, X_3, \ldots X_n, \ldots$ be countably indeterminates over $K$. Let $T = K[Y, X_1, X_2, X_3, \ldots X_n, \ldots]$ and let $J = (X_ny^n : n \geq 1)T \subseteq T$. Let $R = T/J$. Let $Q$ denote the prime ideal of $R$ generated by all the $x_n$, $n \geq 1$, so that $R/Q \cong K[y]$. Let $y$ be the image of $Y$ in $R$. Show that $\text{Hom}_R(R/Q, R) = 0$, while $\text{Hom}_R((R/Q)_{y}, R_y) \neq 0$. Hence, localization at $W = \{y^n : n \in \mathbb{N}\}$ does not commute with $\text{Hom}_R$ in this instance. [Of course, $R/Q$ is not finitely presented, although it is finitely generated.]

5. Let $R$ be a ring and $M$ an $R$-module.
   (a) Prove that $M$ is flat if and only if $M_{P}$ is $R_{P}$-flat for every prime ideal $P$ of $R$.
   (b) Suppose that $R$ is reduced and 0-dimensional. Prove that every $R$-module is flat.
   (c) Let $R$ be a ring such that every $R$-module is flat.
      (1) Prove that for every element $r \in R$ there exists an element $u \in R$ such that $r = ur^2$. Conclude that $V(r)$ is open as well as closed.
      (2) Prove that $R$ is reduced.
      (3) Prove that the Krull dimension of $R$ is 0.

6. Let $R$ be a commutative ring. Suppose that every local ring $R_P$ of $R$ is an integral domain and that $\text{Spec}(R)$ is connected. Show that if $R$ has only finitely many minimal primes, then $R$ is an integral domain.

**EXTRA CREDIT** Suppose that the ring $R$ satisfies the hypotheses in the second sentence of 6., but that no condition is imposed on the minimal primes of $R$. Must $R$ be an integral domain?