

1. S has the K -basis $\mathcal{B} = \{y^n : n \in \mathbb{N}\} \cup \{x^h y^k : h \geq 1, k \in \mathbb{Z}\}$. $\mathcal{B}_0 = \mathcal{B} - \{1\}$ is a K -basis for $m \subseteq S$, and $x\mathcal{B}_0 = \{xy^n : n \geq 1\} \cup \{x^h y^k : h \geq 2, k \in \mathbb{Z}\}$ is a K -basis for J . Hence, the image \mathcal{B}' of $\{y^n : n \in \mathbb{N}\} \cup \{xy^k : k \leq 0\}$ is a K -basis for R . The image u of x is in this basis and so is not 0. Since $xm \subseteq J$, the annihilator \mathfrak{A} of u contains the maximal ideal m/J , and since u is not 0, \mathfrak{A} cannot be larger. Any element of R can be represented uniquely as the image of $f + xg$ where f is a polynomial in y and g is a polynomial in y^{-1} . If f is not zero, let cy^k be the smallest degree term that occurs, where $c \in K - \{0\}$. Multiplying by $c^{-1}x/y^k$ yields u , the image of x . If f is 0 and g is not 0, let cxy^{-k} be the term where k is largest, where $c \in K - \{0\}$. Multiplying by $c^{-1}y^k$ yields u . Thus, u is in every nonzero ideal of R . The K -span of the image \mathcal{S} of $\{xy^k : k \leq 0\}$ (note that $\mathcal{S} \subseteq \mathcal{B}'$) is an ideal P of R , since the product of any element of \mathcal{S} with any element of \mathcal{B}' is either in \mathcal{S} or else 0, and $R/P = K[y]$, as required. \square

2. Suppose that $x, y \in R - \{0\}$ and $xy = 0$. f is in the maximal ideal m of R , and there is a greatest integer $h \geq 0$ such that $x \in f^h R$, since x cannot be in every power of m . Similarly, there is a greatest integer $k \geq 0$ such that $y \in f^k R$. Then $x = f^h u$ with $u \notin fR$ and $y = f^k v$ with $v \notin fR$. Since $xy = 0$, we have that $f^{m+n}uv = 0$, and since f is not a zerodivisor, $uv = 0$. In particular, $uv \in fR$, which is prime. This is a contradiction, since R/fR is a domain. \square

3. (a) After applying $R/J \otimes_R _$, we have $(R/I) \otimes_R J \rightarrow (R/I) \otimes_R R$. The former may be identified with $(R \otimes_R J)/\text{Im}(I \otimes_R J) \cong J/IJ$, and the latter with R/I . An element of J/IJ represented by $j \in J$ maps to 0 in R/I iff $j \in I$, and this occurs if and only if $j \in I \cap J$. Hence, the kernel of the map is $(I \cap J)/IJ$. Part (b) is then immediate from part (a). \square

(c) We must show that if $N \subseteq M$ is injective then $N/JN \rightarrow M/JM$ is injective. Since $N \subseteq M$ is a directed union of inclusions $N_0 \subseteq M_0$ where N_0, M_0 are finitely generated it suffices to consider the case where the modules are finitely generated, and hence we may assume that there are finitely many elements m_1, \dots, m_k such that $M = N + Rm_1 + \dots + Rm_k$. Let $N_i = N + Rm_1 + \dots + Rm_i$, $0 \leq i \leq k$, so that $N_0 = N$ and $N_k = M$. It suffices to show that each of the maps $N_i/JN_i \rightarrow N_{i+1}/JN_{i+1}$ is injective, since we may compose them all. Thus, we have reduced to the case where $M = N + Rm$ is generated over N by one element. Suppose that $u \in N$ represents an element that maps to 0 under $N/JN \rightarrow M/JM$. Then $u \in JM = J(N + Rm) = JN + Jm$, and so $u = v + jm$ where $v \in JN$. We want to show that $u \in JN$. Since $j \in jR \cap J = (jR)J = jJ$, we have $j = jj'$ with $j' \in J$. Then $jm = j'jm = j'(u - v) \in JN$, and $v \in JN$, so that $u = v + jm \in JN$, as required. \square

4. Let x_i be the image of x_i in R . Then R has a K -vector space basis \mathcal{B} consisting of 1 and the monomials $x_1^{a_1} \cdots x_n^{a_n} y^h$ (n may vary) where $a_i, h \in \mathbb{N}$, $a_n \neq 0$, and $h < n$. If $r \in R$ is nonzero and has degree at most n in y when it is written as a K -linear combination of elements of \mathcal{B} , it does not kill x_t for $t \geq n + 1$: in fact x_t does not kill any term of r . Hence, no element of $R - \{0\}$ is killed by Q , which shows that $\text{Hom}_R(R/Q, R) = 0$. In R_y all the X_i map to 0, and $R_y \cong (R/Q)_y \cong K[y, 1/y]$. Thus, $(R/Q)_y \cong R_y$, and $\text{Hom}_{R_y}((R/Q)_y, R_y) \cong R_y \neq 0$. \square

5. (a) We proved in class that flatness is preserved by all base change, so that “only if” is immediate. Now suppose that every M_P is R_P -flat but that $N \subseteq Q$ are R -modules such that $M \otimes_R N \rightarrow M \otimes_R Q$ has a nonzero kernel V . Then we can choose P prime (even maximal) such that $V_P \neq 0$, and (*) $0 \rightarrow V_P \rightarrow (M \otimes_R N)_P \rightarrow (M \otimes_R Q)_P$ is exact. We want to show that for any two R -modules M, N , we have an isomorphism $(M \otimes N)_P \rightarrow M_P \otimes_P N_P$ such that $(m \otimes n)/1$ corresponds to $(m/1) \otimes (n/1)$. Note that for any homomorphism $R \rightarrow S$, if A, B are S -modules, we have a map $A \otimes_R B \rightarrow A \otimes_S B$ that takes $a \otimes_R b$ to $a \otimes_S b$, since $A \times B \rightarrow A \otimes_S B$ mapping $(a, b) \mapsto a \otimes_S b$ is certainly R -bilinear. Thus, we have $M \otimes_R N \rightarrow M_P \otimes_R N_P \rightarrow M_P \otimes_{R_P} N_P$, and by the universal property of base change, this yields a map $(M \otimes_R N)_P \rightarrow M_P \otimes_{R_P} N_P$, since the right hand module is an (R_P) -module. This map sends $m \otimes n \mapsto (m/1) \otimes (n/1)$. To construct the inverse, we note that the map $M_P \times N_P \rightarrow (M \otimes N)_P$ that sends $(m/u, n/v) \mapsto (m \otimes n)/(uv)$ is easily checked to be well-defined (e.g., if $m'/u' = m/u$ then $(m' \otimes n)/(u'v) = (m \otimes n)/(uv)$) and to be (R_P) -bilinear. This map sends $(m/1, n/1) \mapsto (m \otimes n)/1$. Since the two compositions agree with the identity maps on module generators, these maps are mutual inverses. Thus, we have that $M_P \otimes_{R_P} N_P \rightarrow M_P \otimes_{R_P} Q_P$ has nonzero kernel V_P , which contradicts the hypothesis that M_P is (R_P) -flat. \square

(b) Let M be an R -module. If P is any prime of R , R_P is still 0-dimensional and so has a unique prime ideal P . In this ring, every element u/w where $u \in P$ and $w \in R - P$ is nilpotent. But if $(u/w)^n = 0$ then for some $w' \in R - P$, $w'u^n = 0$, and then $(w'u)^n = 0$, so that $w'u = 0$ (R is reduced). Thus, $u/w = 0$, and R_P is reduced. It follows that R_P is a field. Over a field, every module is free and, hence, flat. Therefore every M_P is R_P -flat and we may apply (a). \square

(c) Now assume that every R -module is flat. Given $u \in R$, R/uR is flat and so by **3.**(c), $uR = (uR) \cap (uR) = (uR)(uR) = u^2R$, and so $u \in u^2R$ and $u = ru^2$ for some $r \in R$. Since $u(1 - ru) = 0$, $V(u) \cup V(1 - ru) = \text{Spec}(R)$, while since $(u, 1 - ru) = R$, $V(u) \cap V(1 - ru) = \emptyset$. Thus, $V(u)$ is the complement of the closed set $V(1 - ru)$, and is both open and closed. If R contained a nonzero nilpotent, by repeated squaring one would get an element $u \neq 0$ such that $u^2 = 0$. But then $u = ru^2 = 0$, a contradiction. Hence, R is reduced. If P is any prime ideal of R let $u \in R - P$. Then $u = ru^2$ for some $r \in R$, and so $u(1 - ru) = 0 \in P$. Since $u \notin P$, we have $1 - ru \in P$, and so r represents an inverse for u in R/P . Thus, R/P is a field, and P is maximal. Since this is true for every prime ideal of R , the Krull dimension of R is 0. \square

6. Let P_1, \dots, P_n be the minimal primes of R . Then the sets $V(P_1), \dots, V(P_n)$ are pairwise disjoint, because if Q is a prime containing both P_i and P_j with $P_i \neq P_j$ then the local domain R_Q has at least two minimal primes, a contradiction. Since these closed sets are pairwise disjoint, each is also open: its complement is a finite union of closed sets and therefore closed. Thus, if $n \geq 2$, $\text{Spec}(R)$ is disconnected. Thus, R has a unique minimal prime, which will consist of all the nilpotents in R . But R is reduced, for if $r \neq 0$ is nilpotent, $r/1$ will be nonzero in some R_Q and still nilpotent. Hence, the unique minimal prime is (0) and R is a domain. \square