Math 614, Fall 2008 **Problem Set #5: Solutions**

1. Let $m_1, \dots, m_h$ and $n_1, \dots, n_k$ generate M and N respectively. Then the $m_i \otimes n_j$ generate $M \otimes_R N$, and the map $\mathrm{Hom}_R(M,\, N) \to N^{\oplus h}$ that sends $f \mapsto \bigl(f(m_1), \dots, f(m_h)\bigr)$ is injective, so that $\mathrm{Hom}_R(M,\, N) \hookrightarrow N^h$, which is Noetherian, and so both $M \otimes_R N$ and $\mathrm{Hom}_R(M,\, N)$ are finitely generated. A finitely generated module Q over a Noetherian ring R has finite length iff $\mathrm{Ann}_R(Q)$ is contained only in maximal ideals, by a class result. Both $M \otimes_R N$ and $\mathrm{Hom}_R(M,\, N)$ are kileld by both $\mathrm{Ann}_R M$ and $\mathrm{Ann}_R N$. Hence, if *either* M or N has finite length and the other is finitely generated, the modules $M \otimes_R N$ and $\mathrm{Hom}_R(M,\, N)$ have finite length. $\square$

Note that the coset $C = m + Q$ determines Q as $\{c - c' : c \in C\}$. It follows that if the cosets $C_n = m_n + Q_n$ are desecending, so is the sequence of modules Q_n. Hence, for some N all of the Q_n are equal for $n \geq N$. If two cosets of the same module have nonempty intersection, they are equal. Hence, if one is contained in the other they are equal. It follows that the descending chain of cosets is eventually stable. $\square$

2. (a) Each element of N is killed by P, by definition. If some element $v \neq 0$ had annihilator strictly larger than P, it could be replaced by a nonzero multiple whose annihilator is prime. This would yield an associated prime of N, and, hence, of M that is strictly larger than P, which contradicts the maximality of P in $\mathrm{Ass}\,(M)$. Hence if $n \in N - \{0\}$, $\mathrm{Ann}_{R/P} n = 0$ (i.e., $\mathrm{Ann}_R n = P$) and N is torsion-free over R/P. This also shows that $\mathrm{Ass}_R(N) = \{P\}$. To see that $\mathrm{Ass}\,(M/N) \subseteq \mathrm{Ass}\,(M)$, let $p_1, \dots, p_h$ generate P: the map $M \to M^h$ defined by $m \mapsto (p_1 m, \dots, p_h m)$ has kernel N, yielding an injection $M/N \hookrightarrow M^{\oplus h}$. Hence, $\mathrm{Ass}\,(M/N) \subseteq \mathrm{Ass}\,(M^{\oplus h}) = \mathrm{Ass}\,(M)$. $\square$

(b) Let P be maximal in $\mathrm{Ass}\,(M)$ and let $M_1 = \mathrm{Ann}_R P$. Then $\mathrm{Ass}\,(M/M_1) \subseteq \mathrm{Ass}\,(M)$ and if $M/M_1 \neq 0$ we can repeat the procedure and choose $M_2/M_1 \in M/M_1$ such that it is the annihilator of a maximal element of $\mathrm{Ass}\,(M/M_1) \subseteq \mathrm{Ass}\,(M)$. Recursively, we can choose a strictly ascending sequence $0 \subset M_1 \subset \cdots \subset M_i \subset \cdots$ so that for every i, $\mathrm{Ass}\,(M/M_i) \subseteq \mathrm{Ass}\,(M)$ and M_{i+1}/M_i is the annihilator of a maximal element of $\mathrm{Ass}\,(M/M_i) \subseteq \mathrm{Ass}\,(M)$. By part (a), each M_{i+1}/M_i is torsion-free over a ring of the form R/P_j with $P_j \in \mathrm{Ass}\,(M)$, and the successive quotients continue to have the property that their assassinators are contained in $\mathrm{Ass}\,(M)$. Since M has ACC, the process must eventually stop, which only happens if some M_i is eventually equal to M. Note that $\mathrm{Ass}\,(M) \subseteq \bigcup_i \mathrm{Ass}\,(M_{i+1}/M_i)$, so that every $P_j \in \mathrm{Ass}\,(M)$ must occur. $\square$

3. If $R/P_i \hookrightarrow M$, $S \otimes_R (R/P_i) \hookrightarrow S \otimes M$, and $S \otimes_R (R/P_i) \cong S/P_iS$, so that every $\mathrm{Ass}_S(S/P_iS) \subseteq \mathrm{Ass}_S(S \otimes_R M)$. Thus, $\bigcup_i \mathrm{Ass}_S\bigl(S \otimes_R (R/P_i)\bigr) \subseteq \mathrm{Ass}_S(S \otimes_R M)$. For the other inclusion, choose a filtration $0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$ as problem **2.**. Then we have $0 \subseteq S \otimes_R M_1 \subseteq \cdots \subseteq S \otimes_R M_{n-1} \subseteq S \otimes_R M$ and, consequently, $\mathrm{Ass}_S(S \otimes_R M) \subseteq \bigcup_{i=0}^{n-1} \mathrm{Ass}_S\bigl((S \otimes_R M_{i+1})/(S \otimes_R M_i)\bigr)$. The typical term may be rewritten as $S \otimes_R (M_{i+1}/M_i)$. Hence, it suffices to show that if $A = M_i/M_{i+1}$ is torsion-free over R/P, then $\mathrm{Ass}_S(S \otimes_R A) \subseteq \mathrm{Ass}_S \otimes_R (R/P)$. This will follow if A can be embedded in a finitely generated free (R/P)-module $(R/P)^{\oplus h}$, for then $S \otimes_R A$ embeds in $(S/PS)^{\oplus h}$ whose assassinator is the same as $\mathrm{Ass}\,(S/PS)$. Let $D = R/P$, which is a domain. Let $\mathcal{F}$ be the fraction field of D. Then $\mathcal{F} \otimes_D A$ is a finite-dimensional vector space over $\mathcal{F}$ spanned by the elements of D. Hence, we can choose a basis $d_1, \dots, d_h$ consisting of elements of D.

Then each element a of A is a linear combination of $d_1, \dots, d_h$ with fractional coefficients. It follows that if $a_1, \dots, a_k$ generate A, for every a_j we can choose $b_j \in A - \{0\}$ such that $b_j a_j \in Da_1 + \cdots + Da_h$. Let b be the product of the b_j. Then $A \cong bA \subseteq Aa_1 + \cdots + Aa_h$, which is A-free, and this shows that A embeds in $D^{\oplus h}$, as required. $\square$
For polynomial rings S, $S/PS \cong (R/P)[x_1, \dots, x_n]$ which is a domain if R/P is a domain, so that if P is prime, so is PS and $\mathrm{Ass}_S(S/PS) = \{PS\}$.

4. (a) It suffices to show that for every integer n the natural map $R/m^n \to R_m/\mathcal{M}^n$ is an isomorphism. Because $V(m^n) = m$, R/m^n has the unique maximal ideal m/m^n, and so the image of $R - m$ consists of units. Hence $R_m/\mathcal{M}^n \cong (R/m^n)_m \cong R/m^n$. Taking inverse limits produces the required isomorphism of completions.

(b) In $K[x, y]$ the polynomial $y^2 - x^2 - x^3$ cannot factor, since the only possible factorization even over $K(x)[y]$ is $(y - r)(y + r)$, where r is a square root in$K(x)$ of $x^2 + x^3 = x^2(1 + x)$. If $x^2(1 + x)$ has a square root, so does $1 + x$. Since $K[x]$ is normal, this square root must be in $K[x]$. But the square of any polynomial in $K[x]$ has even degree.

T/fT is not a domain because $1 + x$ does have a square root in $K[[x]]$ when K has characteristic 0 and, more generally, when K has characteristic different from 2. One may use Newton's binomial formula (or Taylor's formula) for $(1 + x)^{1/2} = 1 + \frac{1}{2}x + \cdots + b_n x^n + \cdots$ where $b_n = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-(n-1))}{n!}$. An easy way to see that b_n in lowest terms only has a power of 2 in the denominator is to use the fact that the coefficient of x^n in $(\sum_{n=0}^{\infty} b_n x^n)^2$ is (since $b_0 = 1$) $2b_n + \sum_{j=1}^{n-1} b_j b_{n-j} = 0$ for $n \geq 2$. Hence, $b_1 = 1/2$ and $b_n = (-1/2)\sum_{j-1}^{n-1} b_j b_{n-j}$, which yields the result by induction on n. $\square$

(c) Suppose that f satisfies a nonzero polynomial relation $(*)$ $g_k f^k + \cdots + g_i f^i + \cdots + g_0 = 0$ where $g_k \neq 0$ and the $g_i \in K[x]$. Let d be an upper bound for the degree of g_i if $g_i \neq 0$, and let d_k be the degree of g_k. Choose $n > \max\{k, d, 2\}$. There is a term in $g_k f^k$ of degree $kn! + d_k$ which does not occur elsewhere either in this term or any other term of the expansion of $(*)$ and this yields a contradiction. In fact, every term in f^i has degree which is the sum of i factorials. If all of these factorials are at most $n!$ their sum has degree at most $in!$ and when we multiply by g_i we get degree at most $in! + d_i$ which is too small unless if $i = k$. But even if $i = k$, unless we use the one term that obviously has degree $kn! + d_k$, we get at most $(k - 1)n! + (n - 1)! + d$ which is too small. Therefore, we need only worry about terms which involve $x^{(n+1)!}$ or a higher power of x. But these have too large a degree, since $(n + 1)! > kn! + d_k$.

5. Let $u \in \bigcap_n I_n S$. To show $u \in (y - g)S$ (the other inclusion is clear), work mod $(y - g)S$. The image of u in $A = S/gS$ is in $\bigcap_n (x^n)A = (0)$, since the local ring (A, m_A) is m_A-adically separated, and so $u \in (y - g)S$. Suppose that $F(x, y) \in R$ is a multiple of $y - g$ in S. The $K[x]$-algebra map $K[x, y] \to K[[x]]$ that sends y to g gives a $K[[x]]$-algebra map of completions $K[[x, y]] \to K[[x]]$ that kills $y - g$, and so the map $K[x, y] \to K[[x]]$, kills $F(x, y)$, i.e., $F(x, g) = 0$, a contradiction unless $F = 0$, as required. $\square$

6. (a) Choose $a_0 \in A_0$. Using induction we can construct a sequence $a_n \in A$ such that $a_{n+1} \mapsto a_n$ for all n. At the inductive step, use the fact that $A_{n+1} \to A_n$ is onto to choose a_{n+1} such that $a_{n+1} \mapsto a_n$. $\square$

(b) Let $B_n \subseteq A_n$ denote the constant image of $A_N \to A_n$ for all large N. Since every $A_N \neq \emptyset$, every $B_n \neq \emptyset$. The maps $B_{n+1} \to B_n$ are all onto: given n, choose N so large that the images of A_N in both A_{n+1} and in A_n are both constant. If $a \in A_N$, its image in A_{n+1} is in B_{n+1} and maps to its image in A_n, which is a typical element of B_n. The inverse limit of the B_n is nonempty by (a) and contained in the inverse limit of the A_n. $\square$

(c) It is obvious that an element of the image is in the kernel. Now suppose that $\beta = \{b_n\}_n$ is an element in the inverse image of the B_n that is in the kernel. Let Γ_n denote the inverse image of b_n in A_n, which is nonempty. Then the images of the Γ_N, $N \geq n$, in A_n form a descending sequence of cosets in a module of finite length, which is eventually constant by the last part of **1.**. By (b), $\varprojlim_n \Gamma_n$ is nonempty, and this gives an element of $\varprojlim_n A_n$ that maps to β. $\square$