1. Let $f : R \to S$ be a ring homomorphism, $V$ a multiplicative system in $R$, and $W$ the image of $V$ in $S$.

(a) Explain carefully why there is a unique induced ring homomorphism $g : V^{-1}R \to W^{-1}S$ such that $g(r/1) = f(r)/1$ for all $r \in R$.

(b) Show that if $S$ is module-finite over $R$ (respectively, integral), then $W^{-1}S$ is module-finite (respectively, integral) over $V^{-1}R$.

2. Suppose in 1. (a) that $R \subseteq S$ is a subring (and then $W = V$). Let $T$ be the integral closure of $R$ in $S$. Show that $V^{-1}R \to V^{-1}S$ is injective, and that the integral closure of its image in $V^{-1}S$ is $V^{-1}T$.

3. (a) Which elements in the polynomial ring $K[x, y, z]$ over the field $K$ are integral over $K[x^7, y^{11}, z^{13}]$? Explain your answer.

(b) Let $S$ be the ring of elements in $\mathbb{Q}[\sqrt{11}]$ integral over $\mathbb{Z}$. Show that there is an element $s \in S$ such that $S = \mathbb{Z} + Zs$. Give $s$ explicitly.

4. Let $A \subseteq S$ be rings and let $f, g \in S[x]$ be monic polynomials. Let $R$ be the ring generated over $A$ by the coefficients of the product polynomial $fg$. Show that if $S$ is a domain, then every coefficient of $f$ and of $g$ is integral over $R$. [Suggestion: Enlarge $S$ to an algebraically closed field $L$. Explain why all the roots of $fg$ are integral over $R$. Express the coefficients of $f$ and of $g$ in terms of these roots.]

5. Prove the statement in problem 4. without the assumption that $S$ is a domain.

6. Suppose that $R$ is a principal ideal domain and that $R[z]$, the polynomial ring in one variable over $R$, is isomorphic to $S = K[x, y]$, the polynomial ring in two variables over a field $K$. Prove that $R \cong K[u]$, a polynomial ring in one variable over $K$. [Identify $R$ and $z$ with their images in $S$. Let $u$ be a generator of $m \cap R$, where $m = (x, y)S$. One approach is to prove that every $G \in R$ is in $K[u]$ by induction on the degree of $G$ considered as a polynomial in $S$.]