Math 614, Fall 2008

Supplementary Problem Set #2: Solutions

1. (a) We have a composite map $R \to S \to W^{-1}S$ and the image of V is in the units of $W^{-1}S$. This gives the existence of a unique map as required. Alternatively, since $(r/v)(v/1) = r/1$, we find we must have $g(r/v)g(v/1) = g(r/1)$ and so $g(r/v)(f(v)/1) = f(r)/1$, which implies that $g(r/v) = f(r)/f(v)$. One may define a map on $R \times V$ this way, check that it has a constant value on the equivalence class $[(r, v)]$, and then verify that it is a ring homomorphism. This duplicates work that we did when we proved our representability result characterizing the localization.

(b) If $S = Rs_1 + \cdots + Rs_k$ and $s/f(v) \in W^{-1}S$, then $s = r_1s_1 + \cdots + r_ks_k$ for suitable $r_j \in R$, and then $s/f(v) = \big(f(r_1)/f(v)\big)(s_1/1) + \cdots + \big(f(r_1)/f(v)\big)(s_1/1)$ which we may rewrite as $g(r_1/v)(s_1/1) + \cdots + g(r_k/v)(s_k/1) = (r_1/v)(s_1/1) + \cdots + (r_k/1)(s_k/1)$, because of the way the $V^{-1}R$-module structure is defined on $W^{-1}S$. Thus, $s_1/1, \ldots, s_k/1$ span $W^{-1}S$ as a $V^{-1}R$-module. For the integrality part, note that elements $1/f(v)$ are integral because they are in the image of $V^{-1}R$, while for elements $s/1$ one may take a monic polynomial satisfied by s over R and then $s/1$ satisfies the same polynomial, with the coefficients replaced by their images in $V^{-1}R$. $\square$

2. The element r/v_1 is 0 in $V^{-1}S$ if and only if $r/1 = 0$, since $1/v_1$ is a unit, and this is the case iff $vr = 0$ for some $v \in V$, by a criterion established in class. But since $R \subseteq S$, this criterion also shows that $r/1 = 0$ in $V^{-1}R$, and, hence, so is r/v_1. Because of this, we have injections $V^{-1}R \hookrightarrow V^{-1}T \hookrightarrow V^{-1}S$, and we make the obvious identifications so as to think of these as subrings of $V^{-1}S$. $V^{-1}T$ is integral over $V^{-1}R$ by **1.** (b) above. Now suppose that s/v is integral over $V^{-1}R$. We must show that $s/v \in V^{-1}T$. Consider the monic polynomial satisfied by s/v over $V^{-1}R$. We may multiply by an element of V to clear denominators, yielding a polynomial $v_0s^n + r'_{n-1}s^{n-1} + \cdots + r'_0$ in s over R whose image in $V^{-1}S$ is 0. We may multiply by one element of V to get a polynomial $v_1s^n + r_{n-1}s^{n-1} + \cdots + r_0$ in s over R which really is 0. It is no longer monic, but the leading coefficient is in V. Multiply through by v_1^{n-1} and rewrite the result as $(v_1s)^n + r_{n-1}(v_1s)^{n-1} + \cdots + r_1v_1^{n-2}(v_1s) + r_0v_1^{n-1} = 0$. This shows that v_1s is integral over R, and so $v_1s \in T$, and $s/v = v_1s/(v_1v) \in V^{-1}T$. $\square$

3. (a) All elements are integral over the subring specified. It suffices to show this for the generators x, y, z. But if $u^n = r \in R$, the u satisfies the monic equation $z^n - r = 0$ over R, and so is integral over R. Thus, each of x, y, z is integral over $K[x^7, y^{11}, z^{13}]$. Alternatively, one may argue that the finite set of monomials $x^iy^jz^k$ with $0 \leq i \leq 6$, $0 \leq j \leq 10$, and $0 \leq k \leq 12$ spans the larger ring over the smaller (in fact, these monomials are a free basis).

(b) We shall show that we may take $s = \sqrt{11}$, i.e., that $\mathbb{Z}[\sqrt{11}] = \mathbb{Z} + \mathbb{Z}\sqrt{11}$ is integrally closed. Since $\mathbb{Z}$ is normal, an element of $\mathbb{Q}[\sqrt{11}]$ integral over $\mathbb{Z}$ must have minimal monic polynomial over $\mathbb{Q}$ with coefficients in $\mathbb{Z}$. Suppose that $a + b\sqrt{11}$ is integral over $\mathbb{Z}$, with $a, b \in \mathbb{Q}$. If $b = 0$ then $a \in \mathbb{Q}$ is integral over $\mathbb{Z}$ and therefore in $\mathbb{Z}$. Of $a = 0$, $b\sqrt{11}$ integral implies $11b^2$ is an integer, and so b is an integer (with b and, hence, b^2 in lowest terms, one cannot clear a non-trivial square integer denominator by by multiplying by 11). But then a is integral over $\mathbb{Z}$ and is an integer as well, as desired. Now assume a, $b \neq 0$. Then the other root, $a - b\sqrt{11}$ of the minimal polynomial must also be integral over $\mathbb{Z}$. This implies that the sum $2a$ is integral over $\mathbb{Z}$, and so $a = m/2$, $m \in \mathbb{Z}$, and that the difference $2b\sqrt{11}$ is integral over $\mathbb{Z}$. Therefore, $b = n/2$ where n is an integer. But we must also have that the product $(a + b\sqrt{11})(a - b\sqrt{11}) = a^2 - 11b^2 = (m^2 - 11n^2)/4$ is an integer. The residue of a square integer mod 4 is 0 or 1, depending on whether the integer is even or odd. By a case check, $4 \,|\, (m^2 - 11n^2)$ iff m and n are both even. $\square$ [The analysis is the same for $\mathbb{Z}[\sqrt{p}]$ if p is any prime $\equiv 3$ mod 4, while if $p \equiv 1$ mod 4 one finds that one may take $s = (1 + \sqrt{p})/2$.]

4. If either $f = 1$ or $g = 1$ this is obvious. Assume $\deg f$, $\deg g > 0$. The roots of fg are integral over R because fg is monic. The coefficients of f (resp., g) are the elementary symmetric functions of the roots of f (resp., g), and therefore integral over R.

5. First method. One can use exactly the same method as for the domain case if one can show that every ring has an extension in which a given monic polynomial factors into monic linear factors. (Then extend S so that both f and g factor completely.) One can show this by induction on the degree of the monic polynomial. Call the monic polynomial h and let its degree be d. Then enlarge R to $R_1 = R[x]/hR[x]$, which is free over R on the basis consisting of the images of $1, x, \ldots, x^{h-1}$: thus, R injects into this ring. Let θ be the image of x. Then $h(\theta) = 0$ in R_1. Use the division algorithm over R_1 to divide h by the monic polynomial $x - \theta$. One gets $h(x) = q(x)(x - \theta) + \rho$, where $q(x)$ is monic of degree $d - 1$ and ρ is 0 or degree 0, i.e. $\rho \in R$. Substituting $x = \theta$ shows that $\rho = 0$. Therefore, we can factor out $x - \theta$, and by the induction hypothesis we may enlarge R_1 further so that $q(x)$ factors completely into monic linear factors. $\square$

Second method. Suppose that f has degree m and g has degree n. Instead of considering the original problem, consider the "generic" polynomials $F(x) = x^m + Y_{m-1}x^{m-1} + \cdots + Y_0$ and $G(x) = x^n + Z_{n-1}x^{n-1} + \cdots + Z_0$ over $S_0 = \mathbb{Z}[Y_1, \ldots, Y_{m-1}, Z_1, \ldots, Z_{n-1}]$, and take $A = \mathbb{Z}$. By the domain case established in **4.**, the elements Y_i, Z_j satisfy monic polynomials with coefficients in the ring R_0 generated by the coefficients of FG. Now map S_0 to S by sending the indeterminate coefficients of F, G to the corresponding coefficients of f, g, respectively. This homomorphism takes R_0 into R, and the images of the monic polynomials satisfied by the various Y_i and Z_j provide equations of integral dependence on R for the various coefficients of f and g. $\square$

6. As suggested we think of $R[z]$ as equal to $K[x, y]$. First note that K is 0 together with the units of $K[x, y]$. When R is a domain, every unit of $R[z]$ is in R, and so $K \subseteq R$. Now, $(x, y)K[x, y] \cap R$ is a principal ideal in R: call the generator u. If the intersection where 0, then R would embed in $K[x, y]/(x, y) = K$, and since $K \subseteq R$, it would follow that $R = K$. Thus, uR is not zero, and is a prime and, hence, maximal ideal of R. Note that $u \in K[x, y]$ must be transcendental over K, or else it would be in K, and $\deg u \geq 1$ as a polynomial in $K[x, y]$. It suffices to prove that every $G \in R$ is in $K[u]$, for then $R = K[u]$. Use induction on $n = \deg G$ as a polynomial in x and y. If $n = 0$ then $G \in K \subseteq K[u]$. Say $n > 0$ and that all $H \in R$ with $\deg H < n$ are in $K[u]$. Let $c \in K$ be the constant term of G. Then $G - c \in m \cap R = uR$, and we can write $G = c + Hu$, $H \in R$. Since $\deg H + \deg u = \deg(G - c) = \deg G$ as polynomials in $K[x, y]$ we have that $\deg H < \deg G$. Thus, $H \in K[u]$, and $G = c + Hu \in K[u]$. $\square$