Math 614, Fall 2008 **Supplementary Problem Set #4: Solutions**

1. If $u \in \mathrm{frac}(T)$ then $u \in \mathrm{frac}(S_i)$ for all i. If u is integral over T, it is integral over every S_i, and therefore in every S_i. But then $u \in T$. $\square$

2. This ring is not Noetherian. Consider the decreasing sequence of closed subintervals $V_n = [0, 1/n]$, $n \geq 1$, and let I_n be the ideal of all continuous functions vanishing on V_n: this sequence of ideals is clearly non-decreasing. It increases strictly: if $f_n(x) = 0$ for $x \in V_n$ and $f_n(x) = x - (1/n)$ for $x \in [1/n, 1]$, then for all n, $f_{n+1} \in I_{n+1} - I_n$. $\square$

3. (a) If $f(X) = C \cup D$ for proper closed sets C, $D \subseteq f(X)$, then $f^{-1}(C)$, $f^{-1}(D)$ would be proper closed subsets of X' whose union is X': they are closed because f is continuous, and they are proper because if $u = f(x') \notin C$ then $x' \notin f^{-1}(C)$, and similarly for $f^{-1}(D)$. This gives a contradiction. $\square$

(b) A matrix has rank at most t iff the corresponding linear map $K^s \to K^r$ factors through a t-dimensional space $K^s \to V_t \to K^r$: if it factors this way, the image is contained in the image of V_t and so has dimension $\leq t$. If the image W has dimension $\leq t$ we may choose a t-dimensional subspace V_t of K^r with $W \subseteq V_t$ and use this. Since $V_t \cong K^t$, this translates into the required statement about matrices. The required morphism is then given by matrix multiplication: it is a morphism because each entry of the product of two matrices is a polynomial in the entries of the matrices being multiplied. It follows that X_t is irreducible, since every $\mathbb{A}_K^N$ is irreducible (the coordinate ring, a polynomial ring, is a domain). One can conclude that the *radical* of the ideal generated by the minors is prime. (This actually is the same as the ideal generated by the minors, but that is *much* harder to prove: cf. [M. Hochster and J. A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058].)

4. (a) If $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \cdots$ is a non-decreasing sequence of ideals of R, then $I_n S$ is also such a sequence in S, and must stabilize: but if $I_n S = I_{n+1} S = \cdots$ the same holds after contracting to R, and we have $I_n = I_{n+1} = \cdots$.

(b) Let $x = x_1, \ldots, x_n$ and λ, μ, ν be variable monomials. Extend ϕ to a map f from $S[[x]] \to R[[x]]$ by letting $f(\sum_\mu s_\mu \mu) = \sum_\mu \phi(s_\mu)\mu$. It is trivial to check that this map is linear, and fixes every element of $R[[x]]$. We need only check that it is $R[[x]]$-linear. But $f\big((\sum_\mu r_\mu)(\sum_\nu s_\nu \nu)\big) = f\big(\sum_\lambda (\sum_{\mu\nu=\lambda} r_\mu s_\nu)\lambda\big) = \sum_\lambda \phi(\sum_{\mu\nu=\lambda} r_\mu s_\nu)\lambda$ (by the definition of f) $= \sum_\lambda (\sum_{\mu\nu=\lambda} r_\mu \phi(s_\nu))\lambda$ (by the R-linearity of ϕ) $= \big(\sum_\mu r_\mu \mu\big)\big(\sum_\nu \phi(s_\nu)\nu\big)$, and this is $(\sum_\mu r_\mu \mu) f(\sum_\nu s_\nu \nu)$, as required.

(c) Consider an ideal I of the union. IS is finitely generated, and the generators may be chosen from I, and so for j sufficiently large, IS is generated by finitely many elements $i_1, \ldots, i_n$ of I that are in R_j. We claim that $i_1, \ldots, i_n$ generate I. Suppose that $u \in I$ is in R_k, where, without loss of generality, we may take $k \geq j$. It will suffice if we can show that $u \in I_1 = (i_1, \ldots, i_n)R_j$. But $I_1 S = IS$, and so u is in $I_1 S \cap R_j = I_1$.

(d) Any inclusion of vector spaces splits: one may extend a basis for the smaller space to a basis for the larger, and the additional basis vectors needed span the complementary direct summand. Thus, every K_i is a direct summand of L over K_i, and, by part (b), $K_i[[x_1, \ldots, x_n]]$ is a direct summand of $L[[x_1, \ldots, x_n]]$, so that we may apply part (c).

5. No nonzero K-linear combination of x and y is in m^2, so that m/m^2 is a K-vector space with basis consisting of the images $\overline{x}$ and $\overline{y}$ of x and y. Therefore $m/m^2 \otimes_k m/m^2$ has a basis consisting of $\overline{x}\otimes\overline{x}$, $\overline{x}\otimes\overline{y}$, $\overline{y}\otimes\overline{x}$, and $\overline{y}\otimes\overline{y}$, and so $\overline{x}\otimes\overline{y}-\overline{y}\otimes\overline{x} \neq 0$. Since $\epsilon = x\otimes y - y\otimes x$ maps to this element under the obvious surjection $m \otimes m \twoheadrightarrow m/m^2 \otimes_K m/m^2$, ϵ is not 0. But x kills ϵ since $x(x\otimes y - y\otimes x) = (x^2)\otimes y - (xy)\otimes x = x \otimes (xy) - x\otimes (xy) = 0$. By symmetry, y must also kill ϵ, and therefore m does.

Next note that if we map the free module $R^2 \twoheadrightarrow m$ by $(r,\, s) \mapsto rx + sy$, then the kernel is spanned by $(-y,\, x)$: $rx + sy = 0$ implies that $x \mid sy$, and since x is prime and x does not divide y, we can write $s = ax$ for some $a \in R$. But the $rx + axy = 0$, and it follows that $r = -ay$, so that $(rm,\, s) = a(-y,\, x)$. This enables us to identify m with $(Rt \oplus Ru)/R(-yt + xu)$ where t and u are free generators mapping to x and y respectively. We shall think of the second copy of m as $(Rv \oplus Rw)/R(-yv + xw)$, where v maps to x and w maps to y. Then $M \otimes_R M \cong (Rt\otimes v \oplus Rt \otimes w \oplus Ru \otimes v \oplus Ru \otimes w)/N$ where N is spanned by the four elements $-yt\otimes v + xu \otimes v$, $-yt \otimes w + xu\otimes w$, $-yt\otimes v + xt \otimes w - yu \otimes v + xu \otimes w$. If we let $t\otimes v, t\otimes w, u \otimes v$, and $u \otimes w$ correspond, in that order, to the standard generators of R^4, what we have is the quotient of R^4 by the span of the four vectors $(-y, 0, x, 0)$, $(0, -y, 0, x)$, $(-y, x, 0, 0)$, and $(0, 0, -y, x)$. Killing $t \otimes y - u \otimes v$ (which corresponds to $(0, 1, -1, 0)$) has the effect of killing $x\otimes y - y \otimes x$ in $m \otimes m$. Therefore, $m \otimes m/R\epsilon$ may be identified with a quotient of the free module on three generators (they correspond to $t \otimes v$, $t \otimes w$, and $u \otimes w$). We identify this with R^3, and the submodule V that we need to kill is then spanned by $(-y, x, 0)$ and $(0, -y, x)$. Now $m \otimes m/R\epsilon \cong R^3/V$ maps to m^2 in such a way that the respective generators map to x^2, xy, and y^2. Therefore, we are done if we show that $\{(a,\, b,\, c) \in R^3 : ax^2 + bxy + cy^2 = 0\}$ (the relations on x^2, xy, y^2) is spanned by $(-y, x, 0)$ and $(0, -y, x)$. But if $ax^2 + bxy + cy^2 = 0$, we have $y \mid ax^2 \Rightarrow y \mid a$. Suppose that $a = a'y$. Adding $a'(-y, x, 0)$ we get a relation $(0, b_1, c_1)$ on x^2, xy, y^2, and this means that $b_1xy + c_1y^2 = 0$, and so $b_1x + c_1y = 0$. As before, we see that (b_1, c_1) is a multiple of $(-y, x)$, and so $(0, b_1, c_1)$ is a multiple of $(0, -y, x)$. □

<u>Alternate method</u> Every $u \in m\otimes m$ has the form $u = a(x\otimes x) + b(x\otimes y) + c(y \otimes x) + d(y\otimes y)$ for $a, b, c, d \in K[x, y]$. We can write a as $f(x) + \theta y$ where $f(x) \in K[x]$, and since $\theta y(x\otimes x) = \theta x \otimes yx = \theta x(x \otimes y)$, by changing b we may assume that $a \in K[x]$. Similarly, by changing c we may assume that $d \in K[y]$. Now, u is in the kernel iff $ax^2 + (b+c)xy + by^2 = 0$, and since no term of ax^2 involves y and no term of by^2 involves x, we must have $a = 0$, $b = 0$, and thus $c = -b$. But then this typical element of the kernel has the form $b(x\otimes y - y \otimes x)$. □

6. Suppose $uv = 0$ in $L \otimes D$ with $u \neq 0$ and $v \neq 0$. Choose a basis for D over K, and choose $b_1, \ldots, b_r$ from this basis so that u and v can be expressed as L-linear combinations of the $1 \otimes b_i$, and additional basis vectors $b_{r+1}, \ldots, b_n$ so that for $i,\, j \leq r$, we may write $b_ib_j = \sum_{k=1}^{n} c_{ijk}b_k$. Then we can write $u = \sum_{i=1}^{k} x_i(1 \otimes b_i)$ and $v = \sum_{j=1}^{k} y_i(1\otimes b_j)$, where at least one $x_i \neq 0$, say $x_{i_0} \neq 0$, and at least one $y_j \neq 0$, say $y_{j_0} \neq 0$. Then $uv = \sum_{i,j} x_iy_j \otimes (\sum_{k=1}^{n} c_{ijk}b_k) = \sum_{k=1}^{n}(\sum_{i,j} c_{ijk}x_iy_j) \otimes b_k$, and the fact that $uv = 0$ implies that the every $\sum_{i,j} c_{ijk}x_iy_j = 0$. Then the system of equations $\sum_{i,j} c_{ijk}X_iY_j = 0$ together with the equations $X_{i_0}Z = 1$ and $Y_{j_0}Z' = 1$ have a solution in L. By problem **5.** of Problem Set #3, these equations have a solution in the algebraically closed field K, say $X_i = f_i$, $Y_j = g_j$. Then $u' = \sum_{i=1}^{k} f_ib_i$ and $v' = \sum_{j=1}^{k} g_jb_j$ give nonzero elements of D whose product is zero, a contradiction. □