Math 614, Fall 2008 **Supplementary Problem Set #5: Solutions**

1. Compare each of two sets of generators with their union. It thus suffices to do the case where one, $u_1, \dots, u_n$, is contained in the other, $u_1, \dots, u_{n+k}$. By induction on k, we may assume $k = 1$. Let $(*)$ $u_{n+1} = \sum_{i=1}^n a_i u_i$. Let M and M' be the relations on $u_1, \dots, u_n$ and $u_1, \dots, u_{n+1}$, respectively. Let $f : M \hookrightarrow M'$ via $\alpha \mapsto (\alpha, 0)$. If $\sigma = (r_1, \dots, r_{n+1}) \in M'$ substitution using $(*)$ shows $g(\sigma) = (r_1 + a_1 r_{n+1}, \dots, r_n + a_n r_{n+1}) \in M$, and $g : M' \to M$ is easily checked to be R-linear such that $g \circ f = 1_M$. Let $\theta = (-a_1, \dots, -a_n, 1) \in M'$. One sees at once that $\mathrm{Ker}\,(g) = \{r_{n+1}\theta : r_{n+1} \in R\} \cong R$. This shows that $M' \cong M \oplus \mathrm{Ker}\,(g) \cong M \oplus R$, and so M' is finitely generated iff M is. $\square$

2. Let $N_t = \mathrm{Ker}\,(f \circ \cdots \circ f) = \mathrm{Ker}\, f^t$. Then $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$. If $N_1 \neq 0$, then $N_k \neq 0$. Since f is onto, so is f^t. If $N_1 \neq 0$ choose $u \in N_1 - \{0\}$ and $v \in M$ such that $f^t(v) = u$. Then $v \in N_{t+1} - N_t$, so that chain N_t is strictly increasing, a contradiction. $\square$ (Alternate. Given a counterexample, we may localize so as to keep the kernel nonzero and get another over (R, m). If $v \in m^tM - m^{t+1}M$ is killed, the induced map $m^tM/m^{t+1}M$ onto itself is a surjection of a finite-dimensional vector space onto itself but not injective, a contradiction. This implies that the kernel is in all the $m^{t+1}M$ and so zero. $\square$) $\square$

3. (a) Given $g \in G$ acting on R, consider the composite $g : R \to R \subseteq F = \mathrm{frac}(R)$. The image of $R - \{0\}$ consists of units. By the universal property of localization, the composite extends uniquely to F in such a way that $g(r/s) = g(r)/g(s)$. (One may do this from first principles, but then one must check independence of the definition of the choice of equivalence class representative and that one gets a homomorphism: this is re-inventing the wheel.) Also $(gh)(r/s) = (gh)(r)/(gh)(s) = g(h(r))/g(h(s)) = g(h(r)/h(s)) = g(h(r/s)) = (gh)(r/s)$.

(b) $R^G = F^G \cap R$ (an intersection of normal rings is normal; any field is normal).

(c) Let r/s be fixed by G, where $s \in R - \{0\}$. Let $G = \{g_1, \dots, g_n\}$, where $g_1 = e$ is the identity. Let $s' = g_1(s) \cdots g_n(s)$. Then $ss' = g_1(s) \cdots g_n(s)$ is fixed by G, as is $r/s = (rs')/(ss')$, so that $rs' = (r/s)ss'$ is in R^G. Then $r/s = rs'/ss' \in \mathrm{frac}(R^G)$. $\square$

(d) Since $x/y = (\lambda x)/(\lambda y)$ for all λ, $x/y \in F^G$. But R^G is K: every monomial μ of degree d maps to $\lambda^d \mu$ by this action, and so no polynomial f of degree > 0 is fixed by the action: pick a monomial of degree $d > 0$ occurring in f, and then, since K is infinite, we can choose λ not a root of $x^d - 1 = 0$. $\square$

4. (c) $\Rightarrow$ (b) is obvious from the Hilbert basis theorem, and (a) $\Rightarrow$ (b) follows because $R_0 = R/J$ (or because R_0 is a direct summand of R) and J is finitely generated as R is Noetherian. Now suppose that J is finitely generated. Each generator is a sum of homogeneous elements of positive degree, and so it follows that J is finitely generated by homogeneous elements of positive degree, say $f_1, \dots, f_n$. If $R \neq R_0[f_1, \dots, f_n]$, there is a homogeneous element g of R of least degree not in $R_0[f_1, \dots, f_n]$. Then $g \notin R_0$ and so g has positive degree, and is in J. Then we can write $g = \sum_{i=1}^n h_i f_i$, where $h_i \in R$. Each h_i is a sum of homogeneous components: let h_i' be the component of h_i of degree $\deg(g) - \deg(f_i)$. Then we also have $g = \sum_i h_i' f_i$, where $\deg(h_i') < \deg g$, and so every $h_i' \in R_0[f_1, \dots, f_n]$. The result follows. $\square$

5. Let $I = (xy, xz, yz^2, z^3)$. $\mathrm{Rad}\,(I) = (xy, z)R$ and so the minimal primes are (x, z) and (y, z). If we localize at (x, z) then y becomes invertible and $I^{\mathrm{e}} = (x, z^2)^{\mathrm{e}}$. (x, z^2)

is primary in $K[x,z]$, since (x,z) is maximal, and remains primary when we adjoin the indeterminate y. If we localize at (y,z) then x becomes invertible and $I^{\mathrm{ec}} = (y,z)$. These primary components are unique. Ideals J generated by monomials have K-basis $J \cap \mathcal{M}$, where $\mathcal{M}$ is the set of all monomials, and the intersection of such ideals is thus easily seen to be generated by all least common multiples of their monomial generators, one from each ideal. Thus, $(x,z^2) \cap (y,z) = (xy, xz, z^2)$, but $z^2 \notin I$. Intersecting with (x,y,z^3), which is primary to $m = (x,y,z)$, gives the ideal we want. (x,y,z^3) is primary since m is maximal. $I = (x,z^2) \cap (y,z) \cap (x,y,z^3)$ is an irredundant primary decomposition. The only embedded prime is (x,y,z). (x,y,z^3) isn't unique: it can be replaced by any smaller (x,y,z)-primary ideal that contains I, or by $(x, y+z, z^3)$.

6. If not, suppose that $ab = 0$ with $a, b \neq 0$. Then x^n cannot divide a for all n, or else $a \in \cap_n m^n = 0$ by a class theorem, and, similarly, x^n cannot divide b for all n. Write $a = cx^h$ where x does not divide c, and $b = dx^k$, where x does not divide d. Then $x^{h+k}cd = 0$, and since x is not a zerodivisor, we have that $cd = 0$. This gives a contradiction mod x.

EXTRA CREDIT Make M into an $S = R[x]$ module by letting $P(x)$ act the way $P(f)$ does. Now the map is multiplication by $x \in S$. If it is not injective, localize at a prime so that the kernel is still not zero. Then $M = xM$. If x is in the maximal ideal, this contradicts Nakayama's lemma. If not, x is a unit, and the kernel is 0 after all. $\square$

(Without localization, if $u_1, \ldots, u_n$ generate M, let $\underline{u}$ be these generators written as a column vector. There is a size n matrix $A = (s_{ij})$ such that $\underline{u} = Ax\underline{u}$ and it follows as usual (multiply by the classical adjoint of the matrix) that $\det(I - xA)$ kills M. The determinant has the form $1 - rx$, and so $u - xru = 0$ for all $u \in M$. But then if $xu = 0$, $u = rxu = r \cdot 0 = 0$, as required. $\square$)

Alternatively, let $u_1, \ldots, u_n$ generate M, a counterexample. Choose elements $a_{ij}, b_{ij} \in R$ such that for $1 \le i \le n$, $(*)$ $f(u_i) = \sum_j a_{ij} u_j$ (the u_j generate M) and, moreover, $(**)$ $u_i = \sum_j b_{ij} f(u_j)$ (f is onto). Choose $c_1, \ldots, c_n$ such that $v = \sum_i c_i u_i \neq 0$ but $f(v) = 0$. Let R_0 be generated over $\mathrm{Im}\,(\mathbb{Z})$ by the a_{ij}, b_{ij} and c_i. Let M_0 be the R_0-submodule of M generated by $u_1, \ldots, u_n$. By the equations $(*)$, $f(M_0) \subseteq M_0$, by $(**)$, we have that f maps M_0 onto M_0, and $v \in M_0$. This gives a Noetherian counterexample. $\square$

EXTRA CREDIT (a) If x and y are not in the closure of some z, there is no prime contained in both. This means that the product of their complements, which is a multiplicative system, contains 0 (or there would be a prime disjoint from it). Thus, we can find $a \notin x$ and $b \notin y$ such that $ab = 0$. The $x \in D(a)$ and $y \in D(b)$ are disjoint open neighborhoods of x and y. $\square$

(b) We may assume without loss of generality that the ring is reduced. Let $J = \mathrm{Ann}_R a$. Then $aJ = 0 \Rightarrow V(a) \cup V(J) = \mathrm{Spec}\,(R)$. If $V(a) \cap V(J) \cap Y = \emptyset$, then $V(J) \cap Y = D(a) \cap Y$ will be closed, as needed. But if $(a, J) \subseteq P$, a minimal prime, when we localize at P, $a/1 \neq 0$, since $J \subseteq P$. Since R_P has only the prime PR_P, which must be 0 (or it would contain a nonzero nilpotent), this is a contradiction. $\square$