Math 614, Fall 2010
Problem Set #4
Due: Friday, November 19

1. Let $u$ and $v$ be relatively prime elements in the UFD $R$, and let $W$ be an $R$-module.
   Let $J = (u, v)R$, assume $J \neq R$, and let $f : R^2 \rightarrow J$ be such that $f(r_1, r_2) = r_1u + r_2v$.
   (a) Show that the kernel of $f$ is the cyclic module generated by $(v, -u)$.
   (b) Show that $W \otimes_R J \cong (W \oplus W)/\{(uw, -uw) : w \in W\}$.
   (c) Show that $W \otimes_R J \rightarrow W \otimes_R R \cong W$ is injective if and only if whenever $w_1, w_2 \in W$ are
      such that $uw_1 = vw_2$, there exists $w \in W$ such that $w_1 = vw$ and $w_2 = uw$.
   (d) Prove that $J \otimes_R J \rightarrow J$ with $j \otimes j' \mapsto jj'$ is not injective. Hence, $J$ is not $R$-flat.

2. Let $I$ be a directed partially ordered set, and let $\{M_i\}_{i \in I}$ be a direct limit system of $R$-modules. Let $M = \lim_i M_i$ be the direct limit.
   (a) Fix $i \in I$. Show that the modules $N_{ij} = \text{Ker } (M_i \rightarrow M_j)$ form a directed union indexed
      by $\{j \in I : j \geq i\}$.
   (b) Let $N_i = \bigcup_{j \geq i} N_{ij}$. Show that $N_i = \text{Ker } (M_i \rightarrow M)$ for all $i \in I$.
   (c) Show that the modules $\overline{M}_i = M_i/N_i$ form a direct limit system indexed by $I$ with
      maps induced by the maps $M_i \rightarrow M_j$, and that the map $\overline{M}_i \rightarrow \overline{M}_j$ is injective for $j \geq i$.
      Hence, $M$ is the directed union of the isomorphic images of the $\overline{M}_i$ in $M$.

3. Let $M$ be a Noetherian $R$-module and let $f : M \rightarrow M$ be surjective. Prove that $f$ is an isomorphism.

Extra Credit 6 Let $M$ be a finitely generated $R$-module and let $f : M \rightarrow M$ be surjective. Prove that $f$ is an isomorphism.

4. Let $R$ be a principal ideal domain. Let $a, b \in R - \{0\}$ with $\text{GCD}(a, b) = d$.
   (a) Show that $(R/aR) \otimes_R (R/bR) \cong R/dR$.
   (b) Show also that $\text{Hom}_R(R/aR, R/bR) \cong R/dR$.
   (c) Prove that for any two finitely generated torsion modules $M, N$ over $R$, $M \otimes_R N \cong \text{Hom}_R(M, N)$. (This isomorphism is not natural.)

5. Let $R$ be a reduced ring of Krull dimension 0. Prove that for every prime ideal $P$ of $R$, $R_P$ is a field, and that every $R$-module is flat.

6. Recall that in any category $\mathcal{C}$, a morphism $f : X \rightarrow Y$ is an epimorphism if whenever $g, h : Y \rightarrow Z$ and $g \circ f = h \circ f$, then $g = h$. Let $T, S$ be $R$-algebras.
   (a) Show that $R \rightarrow S$ is an epimorphism of rings if and only if the map $S \otimes_R S \rightarrow S$
      induced by the $R$-bilinear map $S \times S \rightarrow S$ such that $(s_1, s_2) \mapsto s_1s_2$ is an isomorphism.
   (b) Prove that if $R$ is a field, an epimorphism $R \rightarrow S$ is an isomorphism or else $S = 0$.
   (c) Prove that if $R \rightarrow S$ is an epimorphism, then so is $T \otimes_R R \rightarrow T \otimes_R S$.
   (d) Prove that if $R \rightarrow S$ is an epimorphism, $P$ is prime in $R$, and $\kappa_P = R_P/PR_P$, then
      $\kappa_P \otimes_R S$ (the scheme-theoretic fiber), as a $\kappa_P$-algebra, is $\kappa_P$ or is 0.