1. Any prime will have to contain at least one variable from each monomial in $I_k$. The minimal primes are therefore generated by subsets of the variables minimal with respect to the property that they contain at least one variable from each consecutive string of $k + 1$ variables mod $n$. These correspond to minimal ascending sequences of indices $1 \leq i_1 < i_2 < \cdots < i_h \leq n$ (the subscripts on the variables occurring in the minimal prime) such that $i_{t+1} - i_t \leq k$ for $1 \leq t \leq h$, where $i_{h+1} = i_1 + n$: these differences give the lengths of the intervals of consecutive variables not used in the minimal prime. “Minimal” means that the condition fails if any $i_t$ is omitted. The Krull dimension of the quotient is therefore $n - h$ where $h$ is the smallest cardinality of a minimal ascending sequence of indices with the required property: this is the largest dimension one gets when one kills a minimal prime. We show that $h = \lceil \frac{n}{k+1} \rceil$. Since each variable occurs in $k + 1$ strings of consecutive variables and there are $n$ such strings, we must use at least $\frac{n}{(k+1)}$ variables to meet all the strings, and, hence, at least $h = \lceil \frac{n}{(k + 1)} \rceil$. But this value of $h$ works: use the $x_{t(k+1)+1}$, $0 \leq t \leq h - 1$. □

2. Let $S = R[y_1, \ldots, y_k]$. By Noether normalization $R$ is module-finite over a ring of the form $K[x_1, \ldots, x_d]$, and so $S$ is module-finite over the ring $A = K[x_1, \ldots, x_d, y_1, \ldots, y_k]$. Then $\dim S = \dim A$, and $A$ is homomorphic image of a polynomial ring in $d + k$ variables and so $\dim A \leq d + k$. □

3. One can write down polynomials of the specified degrees with unknown coefficients. We want to show that one can choose values for these variables in $K$ to make $(\ast)$ hold, knowing that there is a solution in $L$. By equating coefficients of corresponding monomials from the two sides of $(\ast)$ one gets a system of (quadratic) equations in the unknown coefficients variables. For each coefficient variable $Z$ that is not zero in the solution in $L$, one can introduce an additional variable $Z'$ and an additional equation $ZZ' = 1$. This gives a system of polynomial equations with coefficients in $K$ that has a solution in $L$. Write each equation so that the nonzero terms are on the left hand side, so that the equation sets a polynomial $F_i$ over $K$ equal to 0. Since $K$ is algebraically closed, by Hilbert’s Nullstellensatz, if there is no solution in $K$ then the $F_i$ generate the unit ideal. But then there cannot be a solution in $L$ either, a contradiction. Note the new solution over $K$ has exactly the same monomials occurring with nonzero coefficient as the original solution over $L$ did, and this means that degrees are preserved.

4. We show that the transcendence degree of the fraction field, which is the same as the Krull dimension of the domain generated by the monomials, is the same as the rank $r$ of the matrix. Pick $r$ rows that are linearly independent. The corresponding monomials $\mu_1, \ldots, \mu_r$ are algebraically independent, because the $\mu_1^{k_1} \cdots \mu_r^{k_r}$ are mutually distinct as monomials in the $x_j$ and so linearly independent over $K$. For any other monomial $\mu$ corresponding to a row, the vector of exponents is a $\mathbb{Q}$-linear combination of the vectors corresponding to $\mu_1, \ldots, \mu_r$, and we may multiply by a positive integer $N$ to clear denominators. This leads to an equation $\mu^N = \mu_1^{k_1} \cdots \mu_r^{k_r}$ where the $k_j$ are integers, which shows that $\mu$ is algebraic over the field $K(\mu_1, \ldots, \mu_r)$. □
5. (a) If the annihilator \(I\) of \(u\) is not maximal, \(Ru \cong R/I\) with \(u \mapsto 1 + I\), and a maximal ideal \(m \supset I\) different from \(I\) will be a submodule of \(R/I\) that does not contain \(1 + I\), and whose inverse image in \(Ru\) is a submodule of \(Ru\) (and \(M\)) not containing \(u\). \(\square\)

(b) It suffices, given \(u \notin N\), to construct \(W\) with \(u \notin W \supseteq N\) satisfying the stated condition. Use Zorn’s lemma to construct \(W\) maximal with respect to containing \(N\) such that \(u \notin W\). (\(N\) has the required property, and the union of a chain of submodules containing \(N\) and not \(u\) is a submodule that also contains \(N\) and not \(u\).) Then the image of \(u\) is nonzero and is in every submodule of \(M/W\) (if it were not in \(W'/W \neq 0\), then \(u \notin W' \supseteq N\), contradicting the maximality of \(W\)). By part (a), there is a maximal ideal \(m\) that kills \(u + W\) in \(M/W\), and this means that \(m u \subseteq W\). Therefore, \(W\) satisfies the required condition. Alternatively, let \(I\) be the annihilator of the image of \(u\) in \(M/N\), let \(m\) be a maximal ideal of \(R\) containing \(I\), and let \(W = N + m u\). \(\square\)

6. As in the proof of Noether normalization let \(y_i = z_i - z_n^{2i}\), \(1 \leq n \leq n - 1\) and \(y_n = z_n\). (We may use 2 since it exceeds the degree of \(F\) in any \(z_j\).) Then \(z_i = y_i + y_n^{2i}\), \(i < n\) and \(z_n = y_n\). Then \(K[z_1, \ldots, z_n]/(f) \cong K[y_1, \ldots, y_n]/(g)\) where

\[
g = f(y_1 + y_n^2, \ldots, y_{n-1} + y_n^{2n-1}, y_n)
\]

is monic in \(y_n\) over \(K[y_1, \ldots, y_{n-1}]\). Thus, we may take the \(y_i = z_i + z_n^{2i}\), \(1 \leq i \leq n - 1\) as the algebraically independent elements and \(R\) will be module finite over \(K[y_1, \ldots, y_{n-1}]\).

One may instead take \(y_i = x_i + x_n\) for \(i \leq n - 2\) and and \(y_i = x_i\) for \(i \geq n - 2\) and again obtain a polynomial that is monic in \(x_n\). Another solution is to let \(u_i\) be \((-1)^i\) times the coefficient of \(X^{n-i}\) of the polynomial \((X - z_1)\cdots(X - z_n)\). Then all of \(z_1, \ldots, z_n\) are roots of the monic polynomial \(X^n - u_1 X^{n-1} + \cdots + (-1)^n u_n = 0\). Here, \(u_i\) is the sum of the products of \(z_1, \ldots, z_n\) taken \(i\) at a time. It follows that \(K[z_1, \ldots, z_n]\) is module-finite over \(K[u_1, \ldots, u_n]\). Moreover, \(f = u_{n-1}\). Thus, \(K[z_1, \ldots, z_n]/(f)\) is module-finite over the ring generated by the images of \(u_1, \ldots, u_{n-2}, u_n\).

EC 5 First proof. Consider a matrix \(A\) for the map. If it is 0 and the map is injective then \(m = 0\). Otherwise, choose \(t \geq 1\) as large as possible such that some \(t \times t\) submatrix has determinant not 0. By permuting the free bases and, hence, the rows and columns of the matrix we may assume this submatrix is in the upper left corner. If \(m > n \geq t\) we can consider the \(t \times t + 1\) submatrix \(B\) in the upper left corner. Let \(\Delta_i\) be the determinant of the \(t \times t\) submatrix of \(B\) obtained by omitting its \(i\)th column. Let \(v\) be the \(1 \times n\) column vector whose first \(t + 1\) entries are the elements \((-1)^{i-1}\Delta_i\) and whose other entries are 0. Then \(v \neq 0\), but \(A v = 0\): each entry is, up to sign, the expansion by minors with respect to a row of the determinant of a \(t + 1 \times t + 1\) matrix that either has two rows equal or is a submatrix of \(A\). \(\square\)

Second proof. If \(n < m\), extend the map \(A : R^m \to R^n\) to a map \(B : R^m \to R^m\) by composing with the usual inclusion \(R^n \to R^m\). Then \(B\) gives an injective map, but has bottom row 0, and this will also be true for all powers of \(B\). Since a matrix satisfies its characteristic polynomial (one can reduce to the case of a matrix of indeterminates over \(Z\), and since this is a domain one can use the result for its fraction field), one can take a monic polynomial \(x^n + r_1 x^{n-1} + \cdots + r_n I = 0\) of least degree satisfied by \(B\). If \(s \geq 1\), the \((m, m)\)
entry is \( r_s \) on the left and 0 on the right. Hence, \( r_s = 0 \), and \( B(B^{s-1} + \cdots + r_{s-1}I) = 0 \). Since \( B \) is injective, it follows that each column of the second factor is 0, and so we get an equation of smaller degree. When \( s = 1 \) we get the conclusion \( B = 0 \). □

EC 6  (a) Assume that \( K \) is a field. If \( S = K \) the result is clear. Otherwise, \( S \) contains a polynomial \( f \) of positive degree \( d \), and by multiplying by the inverse of the leading coefficient we may assume that \( f \) is monic. Then \( x \) is integral over \( K[f] \), because \( x \) satisfies \( f(z) - f(x) = 0 \), a monic polynomial in \( z \) with coefficients in \( K[f] \): note that \( f(x) \) will be part of the constant term with respect to \( z \). Hence, \( K[x] \) is module-finite over the Noetherian ring \( K[f] \), and so \( S \) must also be module-finite over \( K[f] \), and hence is finitely generated as a \( K \)-algebra.

(b) Consider the subring \( S = \mathbb{Z}[2x^n : n \geq 1] = \mathbb{Z}[2x, 2x^2, 2x^3, \ldots] \). This subring is not finitely generated as a \( \mathbb{Z} \)-algebra and is not Noetherian: in fact, for all \( n \), \( 2x^n \notin (2x^h : 1 \leq h \leq n - 1)S \), because each term when \( \sum_{i=1}^{n-1} F_i 2x^i \) is expanded that involve \( x^n \) involves a term of \( F_i \) of positive degree in \( x \), and this term of \( F_i \) will have even coefficient and will yield a multiple of \( 4x^n \).