1. Consider a system of $m$ linear equations in $n$ variables over a commutative ring $R$, \( \sum_{i=1}^{n} r_{ij} x_i = r_j \), where $1 \leq j \leq m$ indexes the equations and the $r_{ij}, r_j$ are given elements of $R$. Prove that the equations have a solution in $R$ if and only if for every maximal ideal of $R$ the corresponding system in which the coefficients are replaced by their images in $R_m$ has a solution in $R_m$.

2. Let $M$ be a Noetherian $R$-module. Let $u \neq 0$ be an element that is in every nonzero submodule. Prove that there is a maximal ideal $m$ of $R$ and an integer $n > 0$ such that $m^n M = 0$. Also show that $\text{Ann}_M m = \{ v \in M : mv = 0 \}$ is a one-dimensional vector space over $K = R/m$ spanned by $u$.

3. Let $K \subseteq L$ be fields and let $S = L[[x]]$ be the formal power series ring in one variable over $L$. Let $R = K + xL[[x]]$, the subring of $R$ consisting of all power series with constant term in $K$. Prove that $R$ is a Noetherian ring if and only if $L$ is a finite algebraic extension of $K$. Prove that if $L$ is algebraic over $K$ then that $R$ is normal if and only if $K = L$.

4. Identify the vector space of $r \times s$ matrices over the algebraically closed field $K$ with $\mathbb{A}_K^{rs}$ as discussed in class. Let $Z$ denote the closed algebraic set defined by the vanishing of the ideal generated by $(t+1) \times (t+1)$ minors: $Z$ consists of $r \times s$ matrices of rank at most $t$. Prove that $Z$ is irreducible. (Suggestion: show that there is a surjection $\mathbb{A}^{rt+ts}$ onto $Z$, corresponding to the fact that a linear map of rank at most $t$ factors through $K^t$.) What is the dimension of $Z$?

5. Let $R = A[x,y]$ be a polynomial ring and let $J$ be the ideal $xR + yR$. Let $M$ be an $R$-module. Determine the precise conditions on $M$ for the map $J \otimes_R M \to R \otimes_R M \cong M$ to be injective.

6. Let $K$ be an algebraically closed field. Let $S$ denote the set of elements in $K^r \otimes K^s \cong K^{rs}$ that can be written in the form $u \otimes v$ (the set of decomposable tensors). Show that $S$ can be regarded as a closed algebraic set in $\mathbb{A}_K^{rs}$.

**Extra Credit 7** Let $M$ be a finitely generated $R$-module and let let $f : M \to M$ be surjective. Prove that $f$ is an isomorphism.

**Extra Credit 8** Consider an $n \times n$ matrix $M = (A_{ij})$ whose entries consist of $n^2$ mutually commuting $m \times m$ matrices over a commutative ring $R$. Let $D$ be the $m \times m$ matrix obtained by taking the determinant of $M$. $M$ can also be thought of as a block form for an $mn \times mn$ matrix $\mathcal{M}$ over $R$. Show that the determinant of the $mn \times mn$ matrix $\mathcal{M}$ is equal to $\det(D)$. 