1. If \( R \) is a ring, \( f \in R \) and \( I \subseteq R \), \( I :_R f \) denotes the ideal \( \{ r \in R : rf \in I \} \).
   (a) Prove that for each prime \( P \) of \( R \), the image of \( f \) in \( R_P \) is not in \( IR_P \) iff \( P \supseteq I :_R f \).
   (b) Let \( K \) be a field, let \( R = K[x_1, \ldots, x_n] \) be a polynomial ring, let \( I = (x_1^2, \ldots, x_n^2) \) and let \( f = x_1 + \cdots + x_n \). Determine generators for \( I :_R f \) for \( n \leq 4 \). Additional credit will be given for analysis for larger \( n \). (The answer may depend on \( \text{char}(K) \).)

2. Let \( R \) be a nonzero reduced commutative ring with only finitely many prime ideals, all of which are maximal. Show that \( R \) is isomorphic with a finite product of fields.

3. Let \( R \) be a nonzero reduced ring with only finitely many minimal primes. Let \( W \) be the multiplicative system consisting of all elements not in any minimal prime. Show that every element of \( W \) is a nonzerodivisor in \( R \). (Hence, \( R \) injects into \( W^{-1}R \).) Prove that \( W^{-1}R \) is a finite product of fields.

4. (a) Let \( R \) be a ring, \( W \subseteq R \) a multiplicative system, and \( S = W^{-1}R \). Let \( f : M \to N \) be an \( R \)-linear map of \( S \)-modules. Show that \( f \) is \( S \)-linear, i.e., \( \text{Hom}_R(M, N) = \text{Hom}_S(M, N) \).
   (b) Let \( R \) be the polynomial ring \( K[x, y] \) over a field \( K \) and \( S \) be \( K[x, y/x] \) (a subring of the fraction field of \( R \)). Let \( v = y/x \in S \). Note that \( K[x, v] \) is also a polynomial ring in two variables. Let \( M = S/xS \). Is \( \text{Hom}_R(M, S) = \text{Hom}_S(M, S) \)? Prove your answer. [Later EC: Is \( \text{Hom}_R(S, M) = \text{Hom}_S(S, M) \)? In any case, describe both.]

5. If \( P \) is a prime ideal of \( R \), \( P^{(n)} \) denotes the contraction of \( P^nR_P \) to \( R \), and is called the \( n \)th symbolic power of \( P \). Let \( T = K[u, v, w, x, y, z] \) be a polynomial ring over a field \( K \), and let \( f = ux + vy + wz \). Let \( R = T/fT \). Let \( P \) be the ideal of \( R \) generated by \( v, w, x, y, \) and \( z \). Show that \( P \) is prime, and that \( P^{(2)} \neq P^2 \).

6. Let \( R \) be a ring and \( W \subseteq R \) a multiplicative system. Let \( S = W^{-1}R \). Let \( M \) and \( N \) be \( R \)-modules. Note that there is an \( S \)-linear map \( \theta : W^{-1}\text{Hom}_R(M, N) \to \text{Hom}_S(W^{-1}M, W^{-1}N) \) such that \( [f/w] \mapsto (1/w)W^{-1}f \), where \( W^{-1}f \) is as described in class. Show that if \( R = K[x_1, \ldots, x_n, \ldots] \) is the polynomial ring in a countably infinite sequence of variables over a field \( K \), \( W \) is the set of powers of \( x_1 \), \( M = R/I \), where \( I = (x_n : n \geq 2)R \), and \( N = R/J \), where \( J = (x_1^2x_n : n \geq 2)R \), and then the map the map \( \theta \) is not onto: in fact, show that there is an isomorphism \( W^{-1}M \cong W^{-1}N \) that is not in the image of \( \theta \). (Later, we’ll give a condition that is sufficient for \( \theta \) to be an isomorphism.)

Extra Credit 3. Let \( M \) be a module over a ring \( R \). Suppose that \( M_P \) is generated as an \( R_P \)-module by at most one element for every prime \( P \) of \( R \). Must \( M \) be a finitely generated \( R \)-module? Prove your answer.

Extra Credit 4. An integral domain \( R \) is said to be normal if it contains every element \( f \) of its fraction field that satisfies a monic polynomial with coefficients in \( R \). Suppose that \( R \) is a domain that satisfies the weaker condition that whenever \( f \) is in the fraction field and \( f^n \in R \) for some \( n \in \mathbb{Z}_+ \) then \( f \in R \). Must \( R \) be normal? Prove your answer.