1. Let $T = K[x_1, \ldots, x_n]$ be a polynomial ring and let $R = T_f$, where $f = x_1 \cdots x_n$. Determine an explicit $K$-subalgebra $A \subseteq R$ that is generated over $K$ by algebraically independent elements and such that $R$ is module-finite over $A$.

2. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field and let $S$, $T$ be disjoint sets of monomials $x_1^{a_1} \cdots x_n^{a_n}$ in $R$ such that $1 \in S$, $S$ is closed under multiplication, $T$ is closed under multiplication by elements of $S$, and every monomial in $R$ is in $S \cup T$. Let $B = K[S] \subseteq R$. Prove that $B$ is a direct summand of $R$ as a $B$-module by exhibiting a $B$-module complement for $B$ in $R$. (Hence, $B$ is Noetherian and normal.)

3. Let $x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{i,1}, \ldots, x_{i,n_i}, \ldots, x_{k,1}, \ldots, x_{k,n_k}$ be $k$ sequences of mutually distinct indeterminates over a field $K$, such that the $i$th set has $n_i$ elements. Let $R$ be the polynomial ring in all these $x_{ij}$ over $K$ and let $B$ be the $K$-subalgebra of $R$ generated by all monomials of the form $x_{1,t_1}x_{2,t_2} \cdots x_{k,t_k}$, so that there is one factor from each set. Prove that $B$ is a normal Noetherian domain, and determine its Krull dimension.

4. Call an element $P$ of a polynomial ring $R$ over a field special if it has the form $f^7 + g^{11} + h^{13}$, where $f$, $g$, and $h$ are polynomials in $R$. Let $K \subseteq L$ be fields, with $K$ algebraically closed. Suppose that $P$ is in the polynomial ring $K[x_1, \ldots, x_n]$ and is special in $L[x_1, \ldots, x_n]$. Prove that $P$ is special in $K[x_1, \ldots, x_n]$.

5. Let $A$ be an integral domain that is not a field such that intersection of all of its maximal ideals is $(0)$. Let $m$ be a maximal ideal of the polynomial ring $A[x_1, \ldots, x_n]$. Prove that $m$ contains an element of $A - \{0\}$.

6. Let $S = K[x, y, z]$ be a polynomial ring over a field and let $R = K[x + y, xy, xyz] \subseteq S$ ($R$ is also a polynomial ring). Describe explicitly the image of Spec $(S)$ in Spec $(R)$ as a finite union of sets, each of which is the intersection of an open set and a closed set.

**Extra Credit 5.** Let $R = K[x, y]$ and $S = [x, y/x]$ as in 4.(b) of Problem Set #2. Let $M = S/S$. Describe as explicitly as you can $\text{Hom}_R(S, M)$ and $\text{Hom}_S(S, M)$. Are they equal? [Moral: in general, localization behaves better than adjoining a fraction.]

**Extra Credit 6.** Let $R_\lambda$, $\lambda \in \Lambda$, be a colimit system (direct limit system) of principal ideal domains and injective homomorphisms, and let $R$ be the direct limit. Suppose that if $\lambda \leq \mu$ and $r$ is an irreducible element of $R_\lambda$, then its image in $R_\mu$ is irreducible in $R_\mu$. Is $R$ necessarily a principal ideal domain?

**Extra Credit 7.** Let $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \cdots$ be an infinite chain of fields. Let $L$ be their union. Let $R$ denote the subring of the formal power series ring $L[[x]]$ in one variable consisting of all power series $\sum_{n=0}^{\infty} c_n x^n$ such that $c_n \in K_n$ for all $n$. You may assume that $R$ is a ring. Give necessary and sufficient conditions on the chain of fields for $R$ to be Noetherian.